IYGB - SYNOPTIC PAPER P - QUESTION 1

INTEGRATE FIRST

$$\int_k^8 \frac{4}{2x-1}\,dx = \left[4\ln|2x-1| \times \tfrac{1}{2}\right]_k^8 = \left[2\ln(2x-1)\right]_k^8$$

$$= 2\ln 15 - 2\ln(2k-1)$$

NOW SOLVE THE EQUATION

$$\Longrightarrow \int_k^8 \frac{4}{2x-1}\,dx = 1.90038$$

$$\Longrightarrow 2\ln 15 - 2\ln(2k-1) = 1.90038$$

$$\Longrightarrow 2\ln 15 - 1.90038 = 2\ln(2k-1)$$

$$\Longrightarrow \ln(2k-1) = 1.75786\ldots$$

$$\Longrightarrow 2k-1 = e^{1.75786\ldots}$$

$$\Longrightarrow 2k-1 = 5.800013\ldots$$

$$\Longrightarrow k = 3.400006\ldots$$

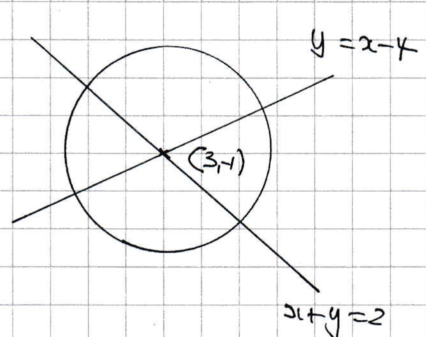
$\therefore k \simeq 3.4$

IYGB - SYNOPTIC PAPER P - QUESTION 2

THE CENTRE OF THE CIRCLE MUST BE THE INTERSECTION OF THESE DIAMETERS

$$\left.\begin{array}{l} y = x-4 \\ x+y=2 \end{array}\right\} \Rightarrow \begin{array}{l} x+(x-4)=2 \\ 2x = 6 \\ x=3 \quad \& \quad y=-1 \end{array}$$

$$\therefore C(3,-1)$$

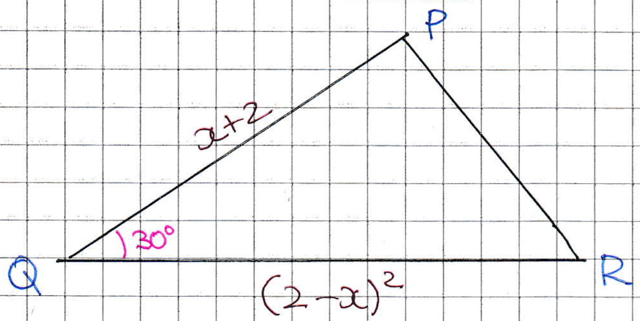


NOW BY INSPECTION THE CIRCLE PASSES THROUGH THE ORIGIN (0,0)

$$(x-3)^2 + (y+1)^2 = r^2$$
$$(0-3)^2 + (0+1)^2 = r^2$$
$$9 + 1 = r^2$$
$$r = \sqrt{10}$$

IYGB – SYNOPTIC PAPER P – QUESTION 3

a) LOOKING AT A DIAGRAM



$$\Rightarrow \text{AREA} = \frac{1}{2}|PQ||QR|\sin 30°$$

$$\Rightarrow A = \frac{1}{2}(x+2)(2-x)^2 \times \frac{1}{2}$$

$$\Rightarrow A = \frac{1}{4}(x+2)(x^2-4x+4)$$

$$\Rightarrow A = \frac{1}{4}\left(x^3-4x^2+4x + 2x^2-8x+8\right)$$

$$\Rightarrow A = \frac{1}{4}(x^3-2x^2-4x+8)$$

// AS REQUIRED

b) DIFFERENTIATING & SOLVING FOR ZERO

$$\Rightarrow \frac{dA}{dx} = \frac{1}{4}\left(3x^2-4x-4\right)$$

$$\Rightarrow 0 = \frac{1}{4}\left(3x^2-4x-4\right)$$

$$\Rightarrow 0 = \frac{1}{4}(3x+2)(x-2)$$

$$\Rightarrow x = \cancel{2},\ -\frac{2}{3}$$

//

CHECKING THE NATURE

$$\frac{d^2A}{dx^2} = \frac{1}{4}(6x-4)$$

$$\left.\frac{d^2A}{dx^2}\right|_{x=-\frac{2}{3}} = \frac{1}{4}\left(6\times\frac{-2}{3}-4\right) = -2 < 0$$

IE A MAX //

IYGB - SYNOPTIC PAPER P - QUESTION 3

FINALLY WE HAVE

$$\Rightarrow A = \frac{1}{4}\left(x^3 - 2x^2 - 4x + 8\right)$$

$$\Rightarrow A_{MAX} = \frac{1}{4}\left(-\frac{8}{27} - \frac{8}{9} + \frac{8}{3} + 8\right)$$

$$\Rightarrow A_{MAX} = 2\left(1 + \frac{1}{3} - \frac{1}{9} - \frac{1}{27}\right)$$

$$\Rightarrow A_{MAX} = 2\left[\frac{27 + 9 - 3 - 1}{27}\right]$$

$$\Rightarrow A_{MAX} = 2 \times \frac{32}{27}$$

$$\Rightarrow A_{MAX} = \frac{64}{27}$$

IYGB - SYNOPTIC PAPER P - QUESTION 4

a) EXPAND BINOMIALLY UP TO x^2

$$f(x) = (1-2x)^{-\frac{1}{2}} = 1 + \frac{-\frac{1}{2}}{1}(-2x)^1 + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{1\times 2}(-2x)^2 + O(x^3)$$

$$= 1 + x + \frac{3}{2}x^2 + O(x^3)$$

b) VALID FOR $|2x| < 1$

$$\Longrightarrow |x| < \frac{1}{2}$$

$$\Longrightarrow -\frac{1}{2} < x < \frac{1}{2}$$

c) LET $x = \frac{1}{8}$

$$(1-2x)^{-\frac{1}{2}} \simeq 1 + x + \frac{3}{2}x^2$$

$$\left(1 - 2\times\frac{1}{8}\right)^{-\frac{1}{2}} \approx 1 + \frac{1}{8} + \frac{3}{2}\left(\frac{1}{8}\right)^2$$

$$\left(1 - \frac{1}{4}\right)^{-\frac{1}{2}} \simeq 1 + \frac{1}{8} + \frac{3}{128}$$

$$\left(\frac{3}{4}\right)^{-\frac{1}{2}} \simeq \frac{147}{128}$$

$$\sqrt{\frac{4}{3}} \simeq \frac{147}{128}$$

$$\frac{2}{\sqrt{3}} \approx \frac{147}{128}$$

$$\frac{\sqrt{3}}{2} \simeq \frac{128}{147}$$

$$\sqrt{3} \simeq \frac{256}{147}$$

IYGB - SYNOPTIC PAPER P - QUESTION 5

a) APPLY THE FACTOR THEOREM

$$f(1)=0 \implies 1^3+a\times1^2+b\times1+c=0$$
$$\implies 1+a+b+c=0$$
$$\implies a+b+c=-1$$

AS REQUIRED

b) APPLYING THE REMAINDER THEOREM TWICE

- $f(2)=-4$

$$\implies 2^3+a\times2^2+b\times2+c=-4$$
$$\implies 8+4a+2b+c=-4$$
$$\implies 4a+2b+c=-12$$

- $f(3)=-12$

$$\implies 3^3+a\times3^2+b\times3+c=-12$$
$$\implies 27+9a+3b+c=-12$$
$$\implies 9a+3b+c=-39$$

USING PART (a) $\implies c=-b-a-1$

$$\implies 4a+2b+(-b-a-1)=-12$$
$$\implies 3a+b=-11$$
$$\implies b=-11-3a$$

$$\implies 9a+3b+(-b-a-1)=-39$$
$$\implies 8a+2b=-38$$
$$\implies 4a+b=-19$$

COMBINE

$$4a+(-11-3a)=-19$$
$$a=-8$$

$$b=-11-3a$$
$$b=-11+24$$
$$b=13$$

$$c=-b-a-1$$
$$c=-13+8-1$$
$$c=-6$$

IYGB - SYNOPTIC PAPER P - QUESTION 5

c) BY LONG DIVISION OR MANIPULATION

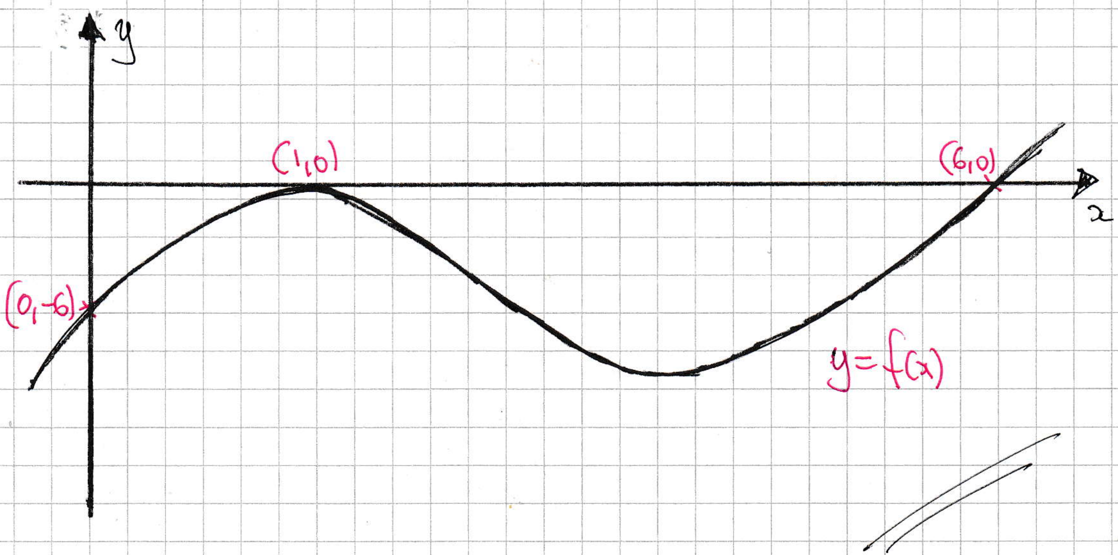
$$f(x) = x^3 - 8x^2 + 13x - 6$$
$$f(x) = x^2(x-1) - 7x(x-1) + 6(x-1)$$
$$f(x) = (x-1)(x^2 - 7x + 6)$$
$$f(x) = (x-1)(x-1)(x-6)$$
$$f(x) = (x-1)^2(x-6)$$

$$\begin{array}{r|l} & x^2 - 7x + 6 \\ \hline x-1 & x^3 - 8x^2 + 13x - 6 \\ & -x^3 + x^2 \\ \hline & -7x^2 + 13x - 6 \\ & +7x^2 - 7x \\ \hline & 6x - 6 \\ & -6x + 6 \\ \hline \end{array}$$

$$\therefore f(x) = (x-1)(x^2-7x+6)$$
$$f(x) = (x-1)(x-1)(x-6)$$

d) COLLECTING ALL THE RESULTS

- $f(x) = +x^3 + \dots$ 
- $x=0 \quad y=-6 \quad \therefore (0,-6)$
- $y=0 \quad x = \begin{cases} 1 & \text{REPEATED} \rightarrow (1,0) \text{ TOUCHING POINT} \\ 6 & (6,0) \text{ CROSSING POINT} \end{cases}$



IYGB - SYNOPTIC PAPER P - QUESTION 6

a) START WITH A GRADIENT CALCULATION FOR $A(1,2)$ & $B(3,8)$.

$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-2}{3-1} = \frac{6}{2} = 3$$

USING THE MIDPOINT OF AB, $M(2,5)$ AND GRADIENT $-\frac{1}{3}$.

$$y - y_0 = m(x - x_0)$$
$$y - 5 = -\frac{1}{3}(x-2)$$
$$3y - 15 = -x + 2$$
$$x + 3y = 17$$

b) BC IS "HORIZONTAL" AS SEEN FROM THE CO-ORDINATES

MIDPOINT IS $\left(\frac{3+13}{2}, \frac{8+8}{2}\right) = (8,8)$

$$\left.\begin{array}{l} \therefore \; l_2 : x = 8 \\ \quad\; l_1 : x + 3y = 17 \end{array}\right\} \Rightarrow 8 + 3y = 17$$
$$\Rightarrow 3y = 9$$
$$\Rightarrow y = 3$$

$\therefore$ $D(8,3)$

COMPUTE 3 DISTANCES DIRECTLY USING $d = \sqrt{(y_2-y_1)^2 + (x_2-x_1)^2}$

$$|AD| = \sqrt{(1-8)^2 + (2-3)^2} = \sqrt{49+1} = \sqrt{50}$$
$$|BD| = \sqrt{(3-8)^2 + (8-3)^2} = \sqrt{25+25} = \sqrt{50}$$
$$|CD| = \sqrt{(13-8)^2 + (8-3)^2} = \sqrt{25+25} = \sqrt{50}$$

INDEED TRUE $|AD| = |BD| = |CD|$

IYGB - SYNOPTIC PAPER P - QUESTION 7

LET THE POSITIVE ODD SQUARE NUMBER BE $(2n+1)^2$, $n \in \mathbb{N}$

$$
\begin{aligned}
(2n+1)^2 - 1 &= 4n^2 + 4n + 1 - 1 \\
&= 4n^2 + 4n \\
&= 4n(n+1)
\end{aligned}
$$

BUT $n(n+1)$ REPRESENTS THE PRODUCT OF 2 CONSECUTIVE INTEGERS, SO IT MUST BE EVEN (ODD × EVEN = EVEN OR EVEN × ODD = EVEN)

$$= 4 \times 2m \quad , m \in \mathbb{N} \qquad 2m = n(n+1)$$

$$= 8m$$

INDEED TRUE

IYGB - SYNOPTIC PAPER P - QUESTION 8

MANIPULATE AS FOLLOWS

$$\Rightarrow \log_a x = \log_{a^2}(x+20)$$

$$\Rightarrow \log_a x = \frac{\log_a (x+20)}{\log_a a^2}$$

$$\Rightarrow \log_a x = \frac{\log_a (x+20)}{2\log_a a}$$

$$\Rightarrow \log_a x = \frac{\log_a (x+20)}{2}$$

$$\Rightarrow 2\log_a x = \log_a (x+20)$$

$$\Rightarrow \log_a x^2 = \log_a (x+20)$$

$$\Rightarrow x^2 = x + 20$$

$$\Rightarrow x^2 - x - 20 = 0$$

$$\Rightarrow (x-5)(x+4) = 0$$

$$x = \begin{cases} 5 \\ \cancel{-4} \end{cases}$$

CHANGE OF BASE

$$\log_a x \equiv \frac{\log_b x}{\log_b a}$$

$\therefore \; x = 5$ //

IYGB - SYNOPTIC PAPER P - QUESTION 9

WRITE EXPLICITLY SOME OF THE TERMS

$$\Longrightarrow \sum_{n=1}^{20}(25+np) = 80$$

$$\Longrightarrow (25+1p)+(25+2p)+(25+3p)+\dots+(25+20p) = 80$$

THE LEFT HAND SIDE IS AN ARITHMETIC PROGRESSION WITH

$a = 25+p$
$d = p$
$L = 25+20p$
$n = 20$

$$S_n = \frac{n}{2}[a+L]$$

$$\Longrightarrow \frac{20}{2}\left[(25+p)+(25+20p)\right] = 80$$

$$\Longrightarrow 10[50+21p] = 80$$

$$\Longrightarrow 50+21p = 8$$

$$\Longrightarrow 21p = -42$$

$$\Longrightarrow p = -2$$

IYGB – SYNOPTIC PAPER P – QUESTION 10

a) THIS IS A QUADRATIC IN e^x

$$\Rightarrow e^{2x} + 2 = 3e^x$$

$$\Rightarrow e^{2x} - 3e^x + 2 = 0$$

$$\Rightarrow (e^x)^2 - 3(e^x) + 2 = 0$$

$$\Rightarrow (e^x - 1)(e^x - 2) = 0$$

$$\Rightarrow e^x = \begin{cases} 1 \\ 2 \end{cases} \qquad \therefore x = \begin{cases} 0 \\ \ln 2 \end{cases}$$

b) REWRITE THE EQUATION AS FOLLOWS

$$\Rightarrow e^{2y-2} + 2 = 3e^{y-1}$$

$$\Rightarrow e^{2y-2} - 3e^{y-1} + 2 = 0$$

$$\Rightarrow e^{2(y-1)} - 3e^{y-1} + 2 = 0$$

$$\Rightarrow e^{2x} - 3e^x + 2 = 0 \qquad \text{WHERE } x = y-1$$

USING PART (a)

$$x = y - 1 = \begin{cases} 0 \\ \ln 2 \end{cases}$$

$$\therefore y = \begin{cases} 1 \\ 1 + \ln 2 \end{cases}$$

IYGB, SYNOPTIC PAPER P, QUESTION 10

c) USING EITHER OF THE APPROACHES SHOWN BELOW

- $e^{t} = 3^{\frac{3}{\ln 3}}$

$\Rightarrow \ln\left(e^{t}\right) = \ln\left(3^{\frac{3}{\ln 3}}\right)$

$\Rightarrow t = \frac{3}{\cancel{\ln 3}} \times \cancel{\ln 3}$

$\Rightarrow t = 3$ //

- $e^{t} = 3^{\frac{3}{\ln 3}}$

$\Rightarrow \left(e^{t}\right)^{\ln 3} = \left(3^{\frac{3}{\cancel{\ln 3}}}\right)^{\cancel{\ln 3}}$

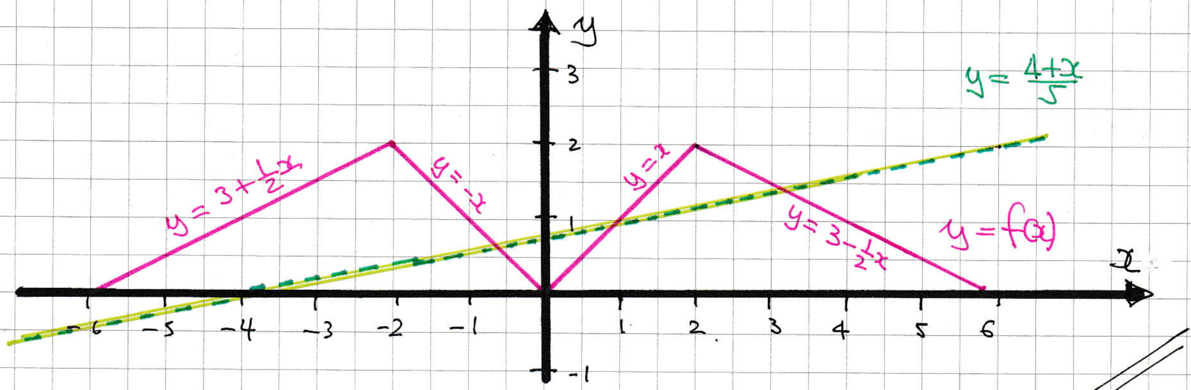
$\Rightarrow \left(e^{\ln 3}\right)^{t} = 3^{3}$

$\Rightarrow 3^{t} = 3^{3}$

$\Rightarrow t = 3$ //

IYGB - SYNOPTIC PAPER P - QUESTION 11

a) AS $f(x)$ IS EVEN, SKETCH BETWEEN 0 & 6 & REFLECT THE GRAPH ABOUT THE y AXIS



b) REARRANGE THE EQUATION TO BE SOLVED

$$\Rightarrow x = 4 + 5f(x)$$
$$\Rightarrow 5f(x) = 4 + x$$
$$\Rightarrow f(x) = \frac{4+x}{5}$$

SKETCHING $y = \frac{x+4}{5}$ IN THE SAME SET OF AXIS (GREEN)

- $\frac{4+x}{5} = -x$
  $4+x = -5x$
  $6x = -4$
  $x = -\frac{2}{3}$

- $\frac{4+x}{5} = x$
  $4+x = 5x$
  $4 = 4x$
  $x = 1$

- $\frac{4+x}{5} = 3 - \frac{1}{2}x \quad \times 10$
  $8+2x = 30-5x$
  $7x = 22$
  $x = \frac{22}{7}$

NOTE THAT $\frac{x+4}{5} = 3 + \frac{1}{2}x$

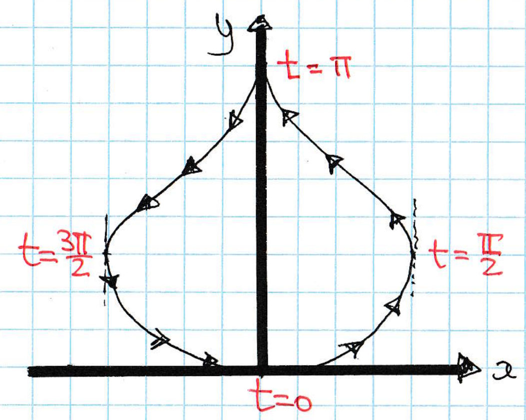
$2x + 8 = 30 + 5x$

$-22 = 3x$

$x = -\frac{22}{3} < -6$ I.E NOT A SOLUTION

IYGB - SYNOPTIC PAPER P - QUESTION 12

START BY "TRACING" THE CURVE



USING SYMMETRY, & INTEGRATING WITH RESPECT TO y BUT IN PARAMETRIC

$$\text{AREA} = 2 \times \int_{y_1}^{y_2} x(y)\,dy = 2\int_{t_1}^{t_2} x(t)\frac{dy}{dt}\,dt$$

$$= 2\int_0^{\pi} (\sin t)(2t)\,dt = \int_0^{\pi} 4t\sin t\,dt$$

AS REQUIRED

PROCEED BY INTEGRATION BY PARTS

$4t$	4
$-\cos t$	$\sin t$

$$\dots = \Big[-4t\cos t\Big]_0^{\pi} - \int_0^{\pi} -4\cos t\,dt$$

$$= \Big[-4t\cos t\Big]_0^{\pi} + \int_0^{\pi} 4\cos t\,dt$$

$$= \Big[4\sin t - 4t\cos t\Big]_0^{\pi}$$

$$= (0 + 4\pi) - (0 - 0)$$

$$= 4\pi$$

IYGB – SYNOPTIC PAPER P – QUESTION 13

METHOD A (PROBABLY THE BEST APPROACH BY CALCULUS)

- $f(x) = k + 12x - 4x^2$

  $f'(x) = 12 - 8x$

- SOLVE FOR ZERO

  $0 = 12 - 8x$

  $8x = 12$

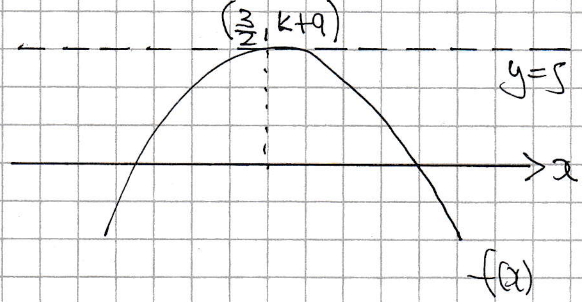
  $x = \frac{3}{2}$

- $f\left(\frac{3}{2}\right) = k + 12\left(\frac{3}{2}\right) - 4\left(\frac{3}{2}\right)^2$

  $f\left(\frac{3}{2}\right) = k + 18 - 9$

  $f\left(\frac{3}{2}\right) = k + 9$

- LOOKING AT A SKETCH OF $f(x)$



- WE REQUIRE THAT $k+9 > 5$

  $\therefore k > -4$

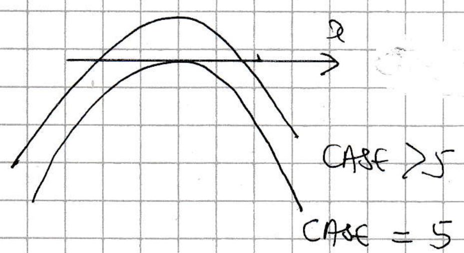
METHOD B (DISCRIMINANT AS SUGGESTED)

- $f(x) > 5$

  $k + 12x - 4x^2 > 5$

  $-4x^2 + 12x + k - 5 > 0$

- LOOKING AT THE SKETCH OF THIS



- THE DISCRIMINANT OF THIS INEQUALITY MUST PRODUCE 2 DISTINCT ROOTS

  (THIS WILL STILL BE THE SAME IF WE MODIFIED THIS TO $4x^2 - 12x - k + 5 < 0$)

- "$b^2 - 4ac$" > 0

  $144 - 4 \times (-4)(k-5) > 0$

  $144 + 16(k-5) > 0$

  $16(k-5) > -144$

  $k - 5 > -9$

  $k > -4$

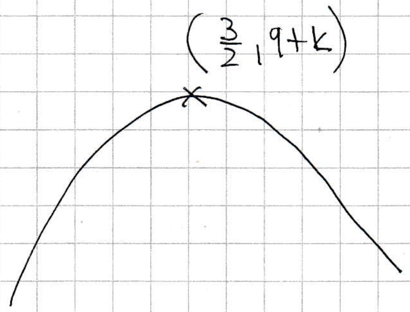
IYGB - SYNOPTIC PAPER P - QUESTION 13

METHOD C - BY COMPLETING THE SQUARE

- $f(x) = k + 12x - 4x^2$

$$f(x) - k = 12x - 4x^2$$
$$-f(x) + k = 4x^2 - 12x$$
$$-f(x) + k = 4\left[x^2 - 3x\right]$$
$$-f(x) + k = 4\left[\left(x - \tfrac{3}{2}\right)^2 - \tfrac{9}{4}\right]$$
$$-f(x) + k = 4\left(x - \tfrac{3}{2}\right)^2 - 9$$
$$f(x) - k = 9 - 4\left(x - \tfrac{3}{2}\right)^2$$
$$f(x) = 9 + k - 4\left(x - \tfrac{3}{2}\right)^2$$

- LOOKING AT A SKETCH



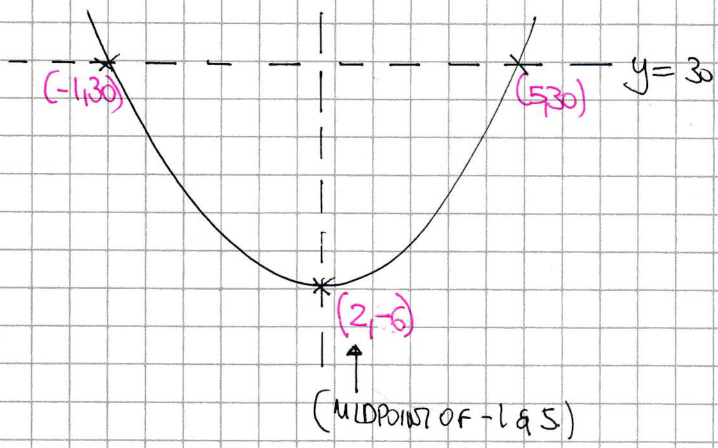
- WE REQUIRE $9 + k > 5$

$$k > -4$$

// AS BEFORE

IYGB - SYNOPTIC PAPER P - QUESTION 18

LOOKING AT SYMMETRIES



WRITE THE FUNCTION IN "COMPLETED THE SQUARE" FORM

$$f(x) = A(x-2)^2 - 6$$

USE (-1,30) OR (5,30) TO EVALUATE THE SCALING CONSTANT A

$$(5,30) \Longrightarrow 30 = A(5-2)^2 - 6$$
$$30 = 9A - 6$$
$$36 = 9A$$
$$A = 4$$

FINALLY SOLVING THE EQUATION $f(x) = 3$

$$\Longrightarrow 4(x-2)^2 - 6 = 3$$
$$\Longrightarrow 4(x-2)^2 = 9$$
$$\Longrightarrow (x-2)^2 = \frac{9}{4}$$
$$\Longrightarrow x - 2 = \begin{cases} \frac{3}{2} \\ -\frac{3}{2} \end{cases}$$
$$\Longrightarrow x = \begin{cases} 7/2 \\ 1/2 \end{cases}$$

$$\therefore x = \frac{7}{2} \cup x = \frac{1}{2}$$

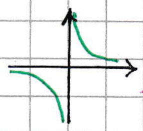
IYGB - SYNOPTIC PAPER P - QUESTION 15

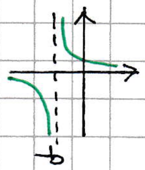
WORKING WITH TRANSFORMATIONS

$\frac{1}{x} \longmapsto \frac{1}{x+b} \longmapsto \frac{1}{(-x)+b} \longmapsto -\frac{1}{b-x}$

REPLACE x FOR $(x+b)$

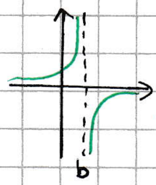
TRANSLATION b UNITS "LEFT"





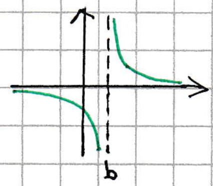
REPLACE x FOR $(-x)$

REFLECT ABOUT THE y AXIS



MULTIPLY THE WHOLE FUNCTION BY -1

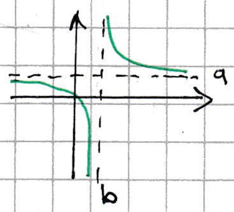
REFLECT ABOUT THE x AXIS



$-\frac{1}{b-x} \longmapsto -\frac{1}{b-x} + a$

ADD a TO THE WHOLE FUNCTION

TRANSLATION "UP" BY a UNITS



NEXT WE NEED THE CORRECT INTERSECTIONS WITH THE AXES

- $x=0$ $\quad y = a - \frac{1}{b}$

  $y = \frac{ab-1}{b} > 0$

  $a>0, b>0, ab>1$

  $\therefore A\left(0, a-\frac{1}{b}\right)$

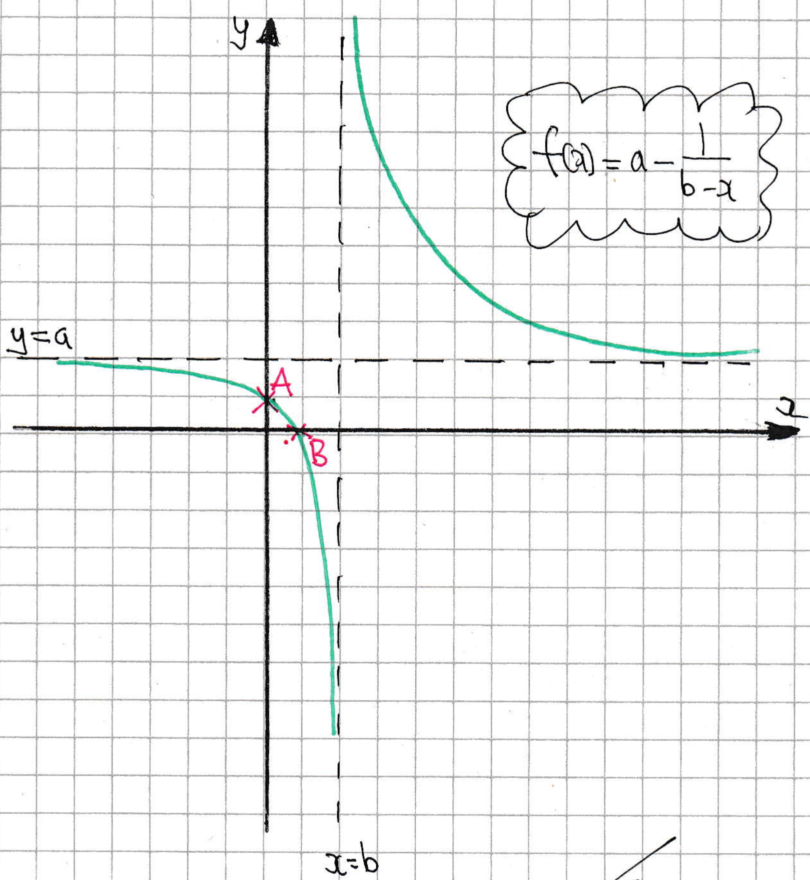
- $y=0$ $\quad 0 = a - \frac{1}{b-x}$

  $\frac{1}{b-x} = a$

  $b - x = \frac{1}{a}$

  $b - \frac{1}{a} = x$

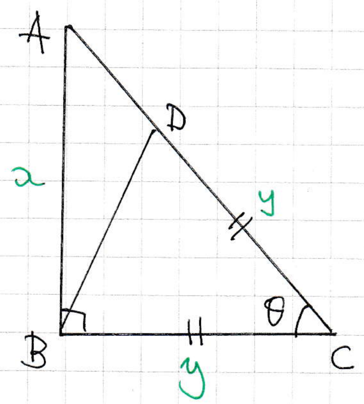
  $\therefore B\left(b-\frac{1}{a}, 0\right)$



$f(x) = a - \frac{1}{b-x}$

IYGB - SYNOPTIC PAPER P - QUESTION 16

STARTING WITH A DIAGRAM



- LET $|AB| = x$ & $|CB| = |CD| = y$
- AREA OF $\overset{\triangle}{ABC} = \frac{1}{2}xy$
- AREA OF $\overset{\triangle}{DBC} = \frac{1}{2}y^2\sin\theta$

NOW WE ARE GIVEN THAT THE AREA OF $\overset{\triangle}{BDC}$ IS 3 TIMES AS LARGE AS THE AREA OF $\overset{\triangle}{ABD}$

$$\Rightarrow \frac{1}{2}y^2\sin\theta = 3\left[\frac{1}{2}xy - \frac{1}{2}y^2\sin\theta\right]$$

$$\Rightarrow \frac{1}{2}y^2\sin\theta = \frac{3}{2}xy - \frac{3}{2}y^2\sin\theta$$

$$\Rightarrow 2y^2\sin\theta = \frac{3}{2}xy$$

$$\Rightarrow 4y^2\sin\theta = 3xy$$

$$\Rightarrow 4y\sin\theta = 3x$$

$$\Rightarrow 4\sin\theta = \frac{3x}{y}$$

BUT $\frac{x}{y} = \tan\theta$

$$\Rightarrow 4\sin\theta = 3\tan\theta$$

// AS REQUIRED

IYGB – SYNOPTIC PAPER P – QUESTION 17

a) USING THE GIVEN IDENTITY

$$\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$$

$$\frac{1}{\tan 3x} = \frac{1 - 3\tan^2 x}{3\tan x - \tan^3 x}$$

$$\cot 3x = \frac{1 - 3\tan^2 x}{3\tan x - \tan^3 x}$$

$$\cot 3x = \frac{1 - \frac{3}{\cot^2 x}}{\frac{3}{\cot x} - \frac{1}{\cot^3 x}}$$

MULTIPLY "TOP & BOTTOM" IN THE R.H.S. BY $\cot^3 x$

$$\cot 3x = \frac{\cot^3 x - 3\cot x}{3\cot^2 x - 1}$$

b) USING THE DOUBLE ANGLE IDENTITIES FOR $\cos 2A$ & $\sin 2A$

$$\text{LHS} = \frac{\cos 2x - \cos x + 1}{\sin 2x - \sin x} = \frac{(2\cos^2 x - 1) - \cos x + 1}{2\sin x\cos x - \sin x}$$

$$= \frac{2\cos^2 x - \cos x}{2\sin x\cos x - \sin x} = \frac{\cos x(\cancel{2\cos x - 1})}{\sin x(\cancel{2\cos x - 1})} \qquad \left(\cos x \neq \frac{1}{2}\right)$$

$$= \frac{\cos x}{\sin x} = \cot x = \text{RHS}$$

AS REQUIRED

IYGB - SYNOPTIC PAPER P - QUESTION 17

c) USING PART (b) — PART (a) IS NOT ACTUALLY NEEDED!

$\Rightarrow \cos 6x + \sin 6x - \cos 3x - \sin 3x + 1 = 0$

$\Rightarrow \cos 6x - \cos 3x + 1 = \sin 3x - \sin 6x$

$\Rightarrow \cos 6x - \cos 3x + 1 = -(\sin 6x - \sin 3x)$

$$\Rightarrow \frac{\cos 6x - \cos 3x + 1}{\sin 6x - \sin 3x} = -1$$

THIS IS THE RESULT OF PART (b) WITH $x \mapsto 3x$

$\Rightarrow \cot 3x = -1$

$\Rightarrow \tan 3x = -1$

$\arctan(-1) = -\frac{\pi}{4}$

$\Rightarrow 3x = -\frac{\pi}{4} \pm n\pi \qquad n = 0, 1, 2, 3, \ldots$

$\Rightarrow x = -\frac{\pi}{12} \pm \frac{n\pi}{3}$

$\therefore \; x_1 = \pi/4$

$x_2 = 7\pi/12$

$x_3 = 11\pi/12$

ALL 3 SOLUTIONS ARE OK

IYGB - SYNOPTIC PAPER P - QUESTION 18

a) USING THE SUBSTITUTION GIVEN

$$\int_1^5 f(x)\,dx = \int_1^3 \frac{2}{x+u}\,(u\,du)$$

$$= \int_1^3 \frac{2u}{x+u}\,du = \int_1^3 \frac{4u}{2x+2u}\,du$$

$$= \int_1^3 \frac{4u}{(u^2+1)+2u}\,du = \int_1^3 \frac{4u}{u^2+2u+1}\,du$$

$$= \int_1^3 \frac{4u}{(u+1)^2}\,du$$ AS REQUIRED

$u = \sqrt{2x-1}$

$u^2 = 2x-1$

$2u\frac{du}{dx} = 2$

$u\frac{du}{dx} = 1$

$dx = u\,du$

$x=1 \longmapsto u=1$

$x=5 \longmapsto u=3$

USING ANOTHER SUBSTITUTION

$$\ldots = \int_2^4 \frac{4u}{v^2}\,dv = \int_2^4 \frac{4(v-1)}{v^2}\,dv$$

$$= 4\int_2^4 \frac{v-1}{v^2}\,dv = 4\int_2^4 \frac{v}{v^2} - \frac{1}{v^2}\,dv$$

$$= 4\int_2^4 \frac{1}{v} - v^{-2}\,dv = 4\Big[\ln|v| + v^{-1}\Big]_2^4$$

$$= 4\Big[\ln|v| + \frac{1}{v}\Big]_2^4 = 4\Big[\left(\ln 4 + \tfrac{1}{4}\right) - \left(\ln 2 + \tfrac{1}{2}\right)\Big]$$

$$= 4\Big[\ln 4 + \tfrac{1}{4} - \ln 2 - \tfrac{1}{2}\Big] = 4\Big[\ln 2 - \tfrac{1}{4}\Big]$$

$$= -1 + 4\ln 2$$

OR $-1 + \ln 16$

$V = u+1$

$\frac{dv}{du} = 1$

$dv = du$

$u=1 \longmapsto v=2$

$u=3 \longmapsto v=4$

IYGB - SYNOPTIC PAPER P - QUESTION 19

START BY FORMING A DIFFERENTIAL EQUATION

$$\frac{dV}{dt} = -k$$

RATE

MELTING/DECREASING

CONSTANT RATE

V = VOLUME OF SNOWBALL (cm^3)
t = TIME (HOURS)

- $t=0$, $r=18$
  $V = \frac{4}{3} \times \pi \times 18^3$
  $V = 7776\pi$

- $t=10$, $r=9$
  $V = \frac{4}{3} \times \pi \times 9^3$
  $V = 972\pi$

SEPARATING VARIABLES

$$\Rightarrow dV = -k\,dt$$

$$\Rightarrow \int 1\,dV = \int -k\,dt$$

$$\Rightarrow V = -kt + C$$

APPLY $t=0$, $V=7776\pi$

$$\Rightarrow 7776\pi = C$$

$$\Rightarrow V = 7776\pi - kt$$

APPLY $t=10$, $V=972\pi$

$$\Rightarrow 972\pi = -10k + 7776\pi$$

$$\Rightarrow 10k = 6804\pi$$

$$\Rightarrow k = 680.4\pi$$

IYGB - SYNOPTIC PAPER F - QUESTION 19

$$\Longrightarrow V = 7776\pi - 680.4\pi t$$

$$\Longrightarrow V = 97.2\pi\left[80 - 7t\right]$$

AS REQUIRED

FINALLY WHEN $r = 4.5$ cm

- $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi (4.5)^3 = 121.5\pi$ cm^3

$$\Longrightarrow 121.5\pi = 97.2\pi\,(80 - 7t)$$

$$\Longrightarrow 1.25 = 80 - 7t$$

$$\Longrightarrow 7t = 78.75$$

$$\Longrightarrow t = 11.25 \text{ hours}$$

IYGB - SYNOPTIC PAPER P - QUESTION 20

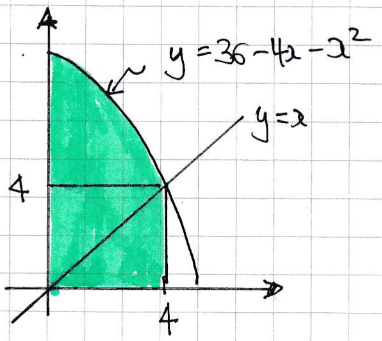
a) SWAPPING x & y YIELDS

$$x = 36 - 4y - y^2$$

b) START BY FINDING THE INTERSECTION OF THE TWO GRAPHS, ON THE LINE $y = x$, WITH POSITIVE CO-ORDINATES

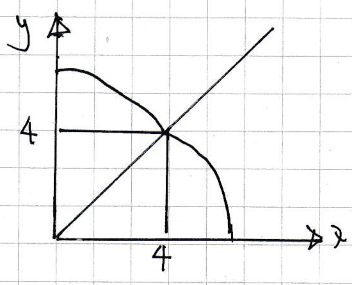
$$\Rightarrow y = 36 - 4x - x^2 \quad \& \quad y = x$$
$$\Rightarrow x = 36 - 4x - x^2$$
$$\Rightarrow x^2 + 5x - 36 = 0$$
$$\Rightarrow (x-4)(x+9) = 0$$
$$\Rightarrow x = \begin{cases} 4 \\ -9 \end{cases} \qquad y = \begin{cases} 4 \\ -9 \end{cases} \qquad \therefore (4,4)$$

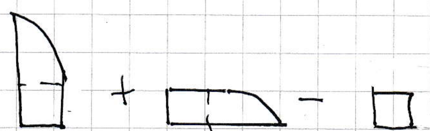
LOOKING AT THE DIAGRAM BELOW THE "GREEN" AREA CAN BE FOUND



$$\int_0^4 36 - 4x - x^2 \, dx = \left[36x - 2x^2 - \tfrac{1}{3}x^3\right]_0^4$$
$$= 144 - 32 - \frac{64}{3}$$
$$= \frac{272}{3}$$

THE REFLECTED PART MUST ALSO HAVE AREA $\frac{272}{3}$, BUT THEN THE 4x4 SQUARE MUST BE SUBTRACTED AS IT IS COUNTED TWICE





$$\frac{272}{3} + \frac{272}{3} - 16 = \frac{496}{3}$$

-1-

IYGB - SYNOPTIC PAPER P - QUESTION 21

WRITE IN EXPLICIT FORM

$$\Rightarrow \sum_{r=2}^{\infty}\left(2^{x-r}\right) = \sqrt{1+3\times 2^{x-2}}$$

$$\Rightarrow 2^{x-2} + 2^{x-3} + 2^{x-3} + 2^{x-4} + \dots = \sqrt{1+3\times 2^{x-2}}$$

USING $S_{\infty} = \frac{a}{1-r}$ WITH $a = 2^{x-2}$ & $r = \frac{1}{2}$

$$\Rightarrow \frac{2^{x-2}}{1-\frac{1}{2}} = \sqrt{1+3\times 2^{x-2}}$$

$$\Rightarrow 2\times 2^{x-2} = \sqrt{1+3\times 2^{x-2}}$$

SQUARING BOTH SIDES

$$\Rightarrow 4\times\left(2^{x-2}\right)^2 = 1+3\times 2^{x-2}$$

$$\Rightarrow 2^2\times 2^{2x-4} = 1+3\times 2^{x-2}$$

$$\Rightarrow 2^{2x-2} = 1+3\times 2^{x-2}$$

MULTIPLY BOTH SIDES BY $4 = 2^2$

$$\Rightarrow 2^{2x} = 4+3\times 2^{x}$$

$$\Rightarrow \left(2^{x}\right)^2 - 3\left(2^{x}\right) - 4 = 0$$

$$\Rightarrow \left(2^{x}+1\right)\left(2^{x}-4\right) = 0$$

$$\Rightarrow 2^{x} = \begin{cases} \cancel{-1} \\ 4 \end{cases}$$

$\therefore$ $x = 2$

IYGB MP2 PAPER P - QUESTION 22

a) BY IMPLICIT DIFFERENTIATION OR REARRANGING AND USING THE QUOTIENT RULE

$$xy = e^x \implies y = \frac{e^x}{x}$$

$$\implies \frac{dy}{dx} = \frac{xe^x - e^x \times 1}{x^2} = \frac{e^x(x-1)}{x^2}$$

WHEN $x=2$, $y = \frac{e^2}{2} = \frac{1}{2}e^2$ & $\frac{dy}{dx} = \frac{1}{4}e^2$

$$\implies y - y_0 = m(x - x_0)$$

$$\implies y - \tfrac{1}{2}e^2 = \tfrac{1}{4}e^2(x-2)$$

$$\implies y - \cancel{\tfrac{1}{2}e^2} = \tfrac{1}{4}e^2x - \cancel{\tfrac{1}{2}e^2}$$

$$\implies y = \tfrac{1}{4}e^2x$$

b) SOLVING SIMULTANEOUSLY "TANGENT" & CURVE

$$\left.\begin{array}{l} xy = e^x \\ y = \frac{1}{4}e^2x \end{array}\right\} \implies x\left(\tfrac{1}{4}e^2x\right) = e^x$$

$$\implies \tfrac{1}{4}x^2e^2 = e^x$$

$$\implies x^2e^2 = 4e^x$$

$$\implies x^2e^2 - 4e^x = 0$$

LET $f(x) = x^2e^2 - 4e^x$

$f(-0.65) = 1.03369\ldots > 0$

$f(-0.55) = -0.0726\ldots < 0$

AS $f(x)$ IS CONTINUOUS, AND CHANGES SIGN IN THE INTERVAL $[-0.65, -0.55]$, THERE EXISTS AT LEAST A SOLUTION IN THIS INTERVAL

$$\implies -0.65 < x < -0.55$$

$$\implies x = -0.6$$ CORRECT TO 1 s.f.

MGB - MP2 PAPER P - QUESTION 22

c) USING THE "FUNCTION" OF PART (b)

$$f(x) = x^2e^2 - 4e^x$$
$$f'(x) = 2e^2x - 4e^x$$

BY NEWTON RAPHSON

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^2e^2 - 4e^{x_n}}{2e^2x_n - 4e^{x_n}}$$

TIDY IS OPTIONAL AS MODERN CALCULATORS CAN HANDLE THIS

$$x_{n+1} = \frac{2e^2x_n^2 - 4x_ne^{x_n} - x_n^2e^2 + 4e^{x_n}}{2e^2x_n - 4e^{x_n}}$$

$$x_{n+1} = \frac{e^2x_n^2 + 4e^{x_n}(1-x_n)}{2e^2x_n - 4e^{x_n}}$$

- USING $x_1 = -0.6$ WE OBTAIN

$x_2 = -0.55798\ldots$

$x_3 = -0.556929\ldots$

- USING $x_1 = -0.55$ (MORE SENSIBLE)

$x_2 = -0.556957\ldots$

$x_3 = -0.556929\ldots$

$\therefore$ REQUIRED x CO-ORDINATE -0.5569

4 sf