# IYGB

## Special Paper W

## Time: 3 hours 30 minutes

**Candidates may NOT use any calculator.**

**Information for Candidates**

This practice paper follows the Advanced Level Mathematics Core Syllabus.
Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

**Advice to Candidates**

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

**Scoring**

Total Score $= T$, Number of non attempted questions $= N$, Percentage score $= P$.

$P = \frac{1}{2}T + N$ (rounded up to the nearest integer)

Distinction $P \geq 70$, Merit $55 \leq P \leq 69$, Pass $40 \leq P \leq 54$

**Question 1**

The quadratic curve $C$, has equation

$$y = 4x - 2x^2 - \frac{1}{2}kx^2,$$

where $k$ is a non zero constant.

Express $y$ in the form

$$\frac{8}{f(k)} - \frac{1}{2}f(k)\left[x - \frac{4}{f(k)}\right]^2,$$

where $f(k)$ is a function to be found **(7)**

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**Question 2**

An arithmetic series has an even number of terms.

The sum of its odd numbered terms, $u_1 + u_3 + u_5 + u_7 + \ldots$, is $752$.

The sum of its even numbered terms, $u_2 + u_4 + u_6 + u_8 + \ldots$, is $800$.

Given further that the difference between the last and the first term of the series is $93$, use an algebraic method to find the number of terms of the series. **(7)**

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**Question 3**

With respect to a fixed origin $O$, the point $A$ and the point $B$ have position vectors $\mathbf{i} - 7\mathbf{j} + 5\mathbf{k}$ and $-9\mathbf{j} + 6\mathbf{k}$, respectively.

**a)** Find a vector equation of the straight line $l$ which passes through $A$ and $B$. **(2)**

A variable vector is defined as

$$\mathbf{p} = (p+6)\mathbf{i} + (2p+3)\mathbf{j} - p\mathbf{k},$$

where $p$ is a scalar parameter.

**b)** Show that for all values of $p$, the point $P$ with position vector $\mathbf{p}$, lies on $l$. **(3)**

**c)** Determine the value of $p$ for which $\overrightarrow{OP}$ is perpendicular to $l$. **(2)**

**d)** Hence, or otherwise, find the shortest distance of $l$ from the origin $O$. **(3)**

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**Question 4**

Solve the following simultaneous equations.

$$2\sum_{r=0}^{\infty} [\log_2 a]^r = \sum_{k=1}^{\infty} (1+b)^{-k} \quad \text{and} \quad \sum_{k=1}^{1} (1+b)^{-k} - \sum_{r=0}^{1} [\log_2 a]^r = \frac{7}{5}.$$

You may leave the answers as indices in their simplest form, where appropriate. **(9)**

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**Question 5**

Determine the range of the following function

$$f(\theta) \equiv \frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}, \quad \theta \in \mathbb{R}$$ **(7)**

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**Question 6**

A circle $C$ is centred at $(a,a)$ and has radius $a$, where $a$ is a positive constant.

The straight line $L$ has equation

$$4x-3y+4=0.$$

Given that $L$ is tangent to $C$ at the point $P$, determine …

**a)** … an equation of $C$. **(10)**

**b)** … the coordinates of $P$. **(2)**

*You may **not** use a standard formula which determines the shortest distance of a point from a straight line in this question.*

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**Question 7**

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k}[u_i] \quad = \quad u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k.$$

A sequence of numbers, $P(1)$, $P(2)$, $P(3)$ ... $P(n)$ is defined by the equation

$$P(n)=\frac{9}{10}\prod_{r=1}^{n}\left[1+\left[\sum_{k=1}^{r}10^k\right]^{-1}\right].$$

Express $P(n)$ in a simplified form not involving a sigma or product operators. **(7)**

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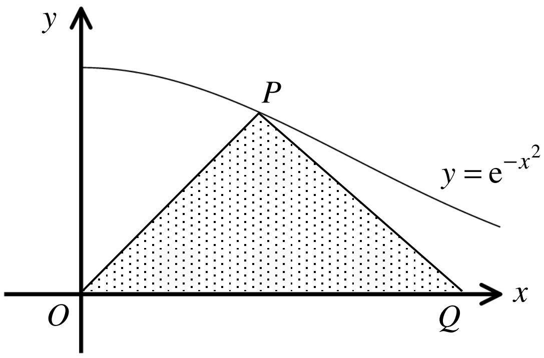
**Question 8**




The figure above shows the graph of the curve with equation

$$y = e^{-x^2}, \quad x \ge 0 .$$

The point $P$ lies on the curve and the point $Q$ lies on the positive $x$ axis so that $|OP| = |PQ|$ where $O$ is the origin.

Show with full justification that the largest area of the triangle $OPQ$ is $\dfrac{1}{\sqrt{2e}}$. **(10)**

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**Question 9**

Given the trigonometric equation

$$\frac{\sin(x-\alpha)}{\cos(x-\alpha) - 2\tan\alpha\sin(x-\alpha)} = \tan\alpha ,$$

show clearly that

$$\tan x = 2\tan\alpha .$$ **(9)**

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**Question 10**

Find in exact simplified form an expression for

$$\int \frac{3x}{x-\sqrt{x^2-1}}\,dx. \qquad (6)$$

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**Question 11**

The real numbers $a$, $b$, $c$ and $d$ satisfy

$$\frac{a}{b}=\frac{c}{d}, \ a \neq b \neq c \neq d \neq 0.$$

**a)** Show that

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}. \qquad (3)$$

**b)** By using the result of part **(a)** or otherwise solve the equation **(9)**

$$\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}=\frac{4x-1}{2}, \ x>1.$$

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**Question 12**

A circle touches the $x$ axis at the origin $O$.

It is further given that the equation of such a circle satisfies the differential equation

$$\left(x^2-y^2\right)\frac{dy}{dx}=y\,f(x),$$

for some function $f$.

Use an algebraic method to find an expression for $f(x)$. **(9)**

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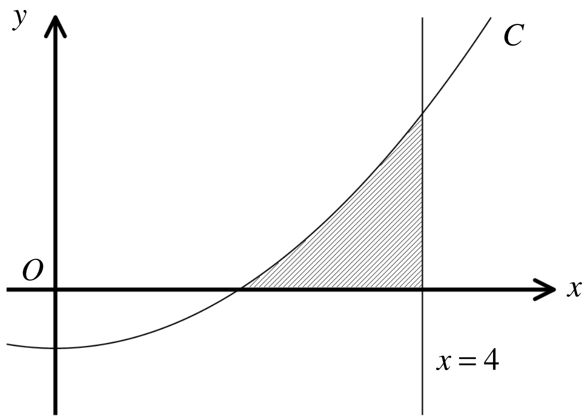
**Question 13**




The figure above shows the curve $C$ with parametric equations

$$x = 2t, \quad y = t^2 - 1, \quad t \in \mathbb{R}.$$

The finite region, bounded by $C$, the $x$ axis and the line $x = 4$ is revolved by $2\pi$ radians about the line $x = 4$, to form a solid of revolution $S$.

Find an exact value for the volume of $S$. **(9)**

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**Question 14**

Solve the following trigonometric equation

$$\sin\left[\operatorname{arccot}(x+1)\right] = \cos(\arctan x).$$

You may assume that $y = \operatorname{arccot} x$ is the inverse function for $y = \cot x$, $0 \le x \le \pi$. **(10)**

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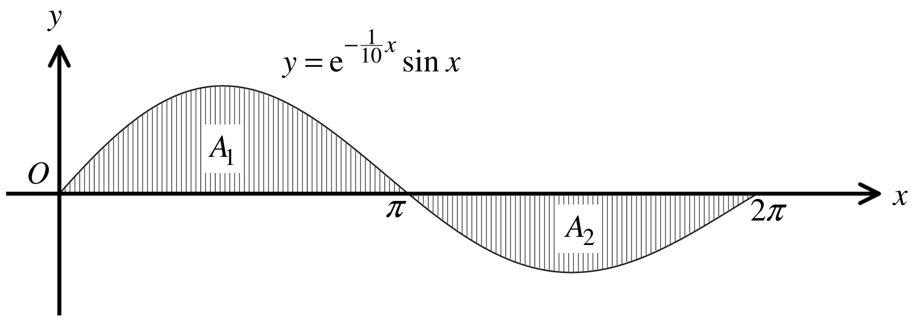
**Question 15**




The figure above shows the curve with equation

$$y = e^{-\frac{1}{10}x} \sin x, \ 0 \le x \le 2\pi .$$

The curve meets the $x$ axis at the origin and at the points $(\pi, 0)$ and $(2\pi, 0)$.

The finite region bounded by the curve for $0 \le x \le \pi$ and the $x$ axis is denoted by $A_1$, and similarly $A_2$ denotes the finite region bounded by the curve for $\pi \le x \le 2\pi$ and the $x$ axis.

By considering a suitable translation of the curve determine with justification the ratio of the areas of $A_1$ and $A_2$. **(8)**

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**Question 16**

Sketch the curve with equation

$$y = \frac{x^2 - 4}{|x+5|}, \ x \in \mathbb{R}, \ x \ne -5 .$$

The sketch must include …

- … the coordinates of all the points where the curve meets the coordinate axes.
- … the equations of the asymptotes of the curve. **(9)**

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**Question 17**

$$f(x) = \begin{cases} x - [x] & x \in \mathbb{R},\ [x] = 2k+1,\ k \in \mathbb{Z} \\ -x + [x] + 1 & x \in \mathbb{R}\ \ [x] = 2k,\ \ k \in \mathbb{Z} \end{cases},$$

where $[x]$ is defined as the greatest integer less or equal to $x$.

Find the value of

$$\frac{\pi^2}{8} \int_{-8}^{8} f(x) \cos(\pi x)\ dx .$$

**(14)**

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**Question 18**

Leibniz rule states that the $n^{\text{th}}$ derivative of the product of the functions $f(x)$ and $g(x)$ satisfies

$$\left[ f(x)\, g(x) \right]^n = \sum_{r=0}^{n} \left[ \binom{n}{r} \left[ \left[ f(x) \right]^{(r)} \left[ f(x) \right]^{(n-r)} \right] \right],$$

where $f^0(x) = f(x)$ , $f^1(x) = f'(x)$, $f^2(x) = f''(x)$, ... , $f^k(x) = \dfrac{d^k}{dx^k}\left[ f(x) \right]$.

Show, by a detailed method, that

$$\frac{d^n}{dx^n}\left[ x^4 \ln x \right] = n!\, x^{4-n} \sum_{r=0}^{4} \left[ \binom{4}{r} f(n,r) \right],$$

where $f(n,r)$ is a function to be found.

**(14)**

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**Question 19**

It is given that

- the angles $A$, $B$ and $C$ are the three angles of a triangle $ABC$.
- the angles $A$, $B$ and $C$ are in an increasing arithmetic progression, in that order.
- The lengths of the triangle $ABC$, opposite each of the angles $A$, $B$ and $C$ are denoted by $a$, $b$ and $c$.

Show that

$$\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A = \sqrt{3}.$$ **(10)**

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**Question 20**

$$f(x) = \frac{1}{\sqrt{1-x}}, \quad -1 < x < 1.$$

**a)** By manipulating the general term of binomial expansion of $f(x)$ show that

$$f(x) = \sum_{r=0}^{\infty} \binom{2r}{r} \left(\tfrac{1}{4}x\right)^r.$$ **(11)**

**b)** Find a similar expression for $\dfrac{1}{\sqrt{16-x^2}}$ and show further that

$$\frac{x}{\left(16-x^2\right)^{\frac{3}{2}}} = \sum_{r=1}^{\infty} \binom{2r}{r} \left(\tfrac{1}{16}r\right)\left(\tfrac{1}{8}x\right)^{2r-1}.$$ **(6)**

**c)** Determine the exact value of

$$\sum_{r=1}^{\infty} \binom{2r}{r} \left(\tfrac{5}{32}r\right)\left(\tfrac{4}{25}\right)^r.$$ **(4)**

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