# IYGB

## Special Paper T

## Time: 3 hours 30 minutes

**Candidates may NOT use any calculator.**

**Information for Candidates**

This practice paper follows the Advanced Level Mathematics Core Syllabus.
Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

**Advice to Candidates**

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

**Scoring**

Total Score $= T$ , Number of non attempted questions $= N$ , Percentage score $= P$ .

$P = \frac{1}{2}T + N$ (rounded up to the nearest integer)

Distinction $P \geq 70$, Merit $55 \leq P \leq 69$, Pass $40 \leq P \leq 54$

**Question 1**

Show by valid mathematical arguments that

$$\sqrt[8]{8!} < \sqrt[9]{9!}.$$ (6)

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**Question 2**

The functions $f$ and $g$ are defined by

$$f(x) \equiv \cos x,\ x \in \mathbb{R},\ 0 \le x \le \pi$$

$$g(x) \equiv 1 - x^2,\ x \in \mathbb{R}.$$ (7)

**a)** Solve the equation

$$fg(x) = \frac{1}{2}.$$

**b)** Determine the range of values of $x$ for which $f^{-1}g(x)$ is **not** defined. (5)

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**Question 3**

The quadrilateral $ABCD$ is a rectangle with the vertex $A$ having coordinates $(2,1,2)$.

The diagonals of the rectangle intersect at the point with coordinates $(7,0,4)$.

**a)** Find the coordinates of the point $C$. (2)

The points $B$ and $D$ both lie on the straight line with vector equation

$$\mathbf{r} = 4\mathbf{i} + 15\mathbf{j} + 10\mathbf{k} + \lambda(\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}),$$

where $\lambda$ is a scalar parameter.

**b)** Determine the coordinates of $B$ and $D$. (6)

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**Question 4**

A circle passes through the points $A(x_1, y_1)$ and $A(x_2, y_2)$.

Given that $AB$ is a diameter of the circle, show that the equation of the circle is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0.$$ **(6)**

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**Question 5**

A curve $C$ has equation

$$\sqrt{y} + \sqrt{x} = 1.$$

Sketch the graph of $C$, for the largest possible domain. **(4)**

The sketch must include, in exact form where appropriate,

- the coordinates of any points where the graph of $C$ meets the coordinate axes.
- the coordinates of any stationary points.
- the coordinates of any non stationary turning points.
- the equations of any asymptotes.

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**Question 6**

The product operator $\prod$, is defined as

$$\prod_{i=1}^{k} [u_i] = u_1 \times u_2 \times u_3 \times u_4 \times \ldots \times u_{k-1} \times u_k .$$

Use a clear method to show that

$$\prod_{m=1}^{3} \prod_{n=1}^{4} \left[\sqrt{mn}\right] = k^3\sqrt{6},$$

where $k$ is a positive integer to be found. **(7)**

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**Question 7**

Show clearly that

$$\frac{d}{dx}\left[\ln\left(x-2+\sqrt{x^2-4x+13}\right)\right]=\frac{1}{\sqrt{x^2-4x+13}}.$$ **(8)**

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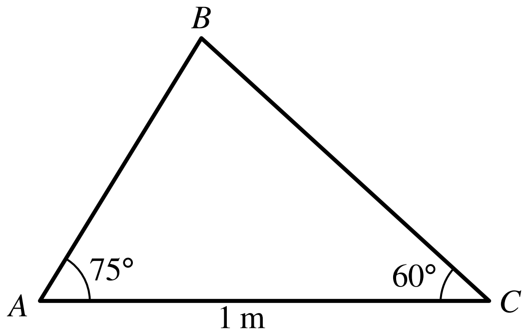
**Question 8**




The figure above shows a triangle $ABC$.

The length of $AC$ is $1$ m. The angles $BAC$ and $BCA$ are $75°$ and $60°$, respectively.

The height of the triangle from the vertex $B$ to the side $AC$ is $h$ cm.

Show that

$$h=\frac{\tan 75°\tan 60°}{\tan 75°+\tan 60°}.$$ **(8)**

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**Question 9**

The $k^{\text{th}}$ of an arithmetic progression is $849$, where $k$ is a positive integer.

The $(k+p)^{\text{th}}$ term and the $(k+2p+1)^{\text{th}}$ term of the same arithmetic progression are $873$ and $905$ respectively, where $p$ is a positive integer.

Find the value of the $(k+20)^{\text{th}}$ term of the progression. **(8)**

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**Question 10**

Solve the trigonometric equation

$$\sqrt{3}\cos\left(x+\frac{\pi}{5}\right)=\sin\left(x+\frac{\pi}{5}\right), \qquad 0 \le \theta < 2\pi,$$

giving the answers in terms of $\pi$. **(10)**

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**Question 11**

Use an algebraic method to show that $x=1$ and $y=-1$ is the only real solution pair of the following simultaneous equations

$$x^4+y^4=2 \quad \text{and} \quad x-y=2.$$ **(10)**

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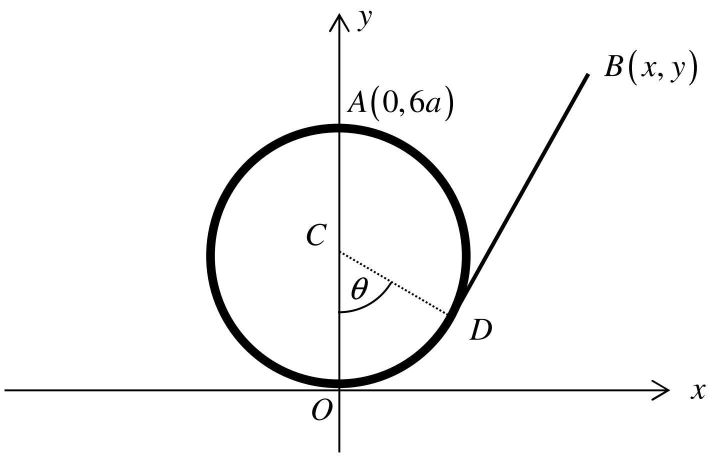
**Question 12**




The figure above shows a set of coordinate axes superimposed with a circular cotton reel of radius $a$ and centre at $C(0,3a)$.

A piece of cotton thread, of length $3\pi a$, is fixed at one end at $O$ and is being unwound from around the circumference of the fixed circular reel. The free end of the cotton thread is marked as the point $B(x,y)$ which was originally at $A(0,6a)$.

The unwound part of the cotton thread $BD$ is kept straight and $\theta$ is the angle $OCD$ as shown in the figure above.

**a)** Determine the parametric equations that satisfy the locus of $B(x,y)$, as the cotton thread is unwound in the fashion described, for which $x>0$, $y>0$. **(4)**

**b)** Find the total area enclosed by the curve traced by $B$, in the entire $x$-$y$ plane. **(16)**

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**Question 13**

$$\sqrt{x+2+\sqrt{x+2+\sqrt{x+2+\sqrt{x+2+\sqrt{x+2+\ldots}}}}},$$

It is given that the above nested radical converges to a limit $L$, $L \in \mathbb{R}$.

Determine the range of possible values of $x$. **(7)**

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**Question 14**

Solve the differential equation

$$\frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right)^2 = 1,$$

given that $y = 0$ and $\frac{dy}{dx} = \frac{1}{6}$ at $x = 0$, giving the answer in the form $y = f(x)$.

**(16)**

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**Question 15**

Solve the following trigonometric equation

$$\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta} = 1, \quad 0 \le \theta < 2\pi.$$

**(10)**

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**Question 16**

$$\sum_{r=1}^{\infty}\left[\frac{1}{r^2}\right] = L.$$

It is given that the above infinite series converges to a limit $L$.

Find, in terms of $L$ where appropriate, the limit of each of the following infinite series.

**a)** $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \ldots$ **(2)**

**b)** $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \ldots$ **(3)**

**c)** $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \ldots$ **(3)**

**d)** $\frac{1}{1^2} + \frac{1}{2^2} - \frac{8}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} - \frac{8}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} - \frac{8}{9^2} + \ldots$ **(4)**

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**Question 17**

$$I = \int \sqrt{\tan x}\ dx .$$

**a)** Use a suitable substitution to show that

$$I = \int \frac{1+\frac{1}{u^2}}{\left(u-\frac{1}{u}\right)^2+2}\ dx \quad + \quad \int \frac{1-\frac{1}{u^2}}{\left(u+\frac{1}{u}\right)^2-2}\ dx . \qquad \textbf{(6)}$$

**b)** By using a further substitution in each of the integrals of part **(a)** find a simplified expression for $I$, in terms of $x$. **(9)**

*You may assume without proof that*

$$\int \frac{1}{x^2+a^2}\ dx = \frac{1}{a}\arctan\left[\frac{x}{a}\right] + \text{constant} .$$

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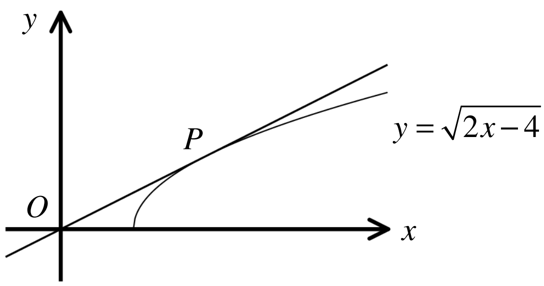
**Question 18**




The figure above shows the graph of the curve $C$ with equation

$$y = \sqrt{2x-4}\ ,\ x \geq 2 .$$

The point $P$ lies on $C$, so that the tangent to the curve at the point $P$ passes through the origin $O$.

Use a calculus method to find the coordinates of $P$. **(9)**

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**Question 19**

Prove that for all $x$ such that $-1 \le x \le 1$

$$\arccos x + \arccos\left[\frac{1}{2}\left(x + \sqrt{3 - 3x^2}\right)\right] = \frac{\pi}{3}. \qquad (12)$$

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**Question 20**

Find as an exact fraction the value of $I$,

$$I = \frac{\int_0^1 \left(1 - x^{20}\right)^{50} \, dx}{\int_0^1 \left(1 - x^{20}\right)^{51} \, dx}. \qquad (12)$$

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