IYGB

Special Paper P

Time: 3 hours 30 minutes

Candidates may NOT use any calculator.

Information for Candidates

This practice paper follows the Advanced Level Mathematics Core Syllabus.
Booklets of *Mathematical formulae and statistical tables* may NOT be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 20 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Scoring

Total Score $= T$, Number of non attempted questions $= N$, Percentage score $= P$.

$P = \frac{1}{2}T + N$ (rounded up to the nearest integer)

Distinction $P \geq 70$, Merit $55 \leq P \leq 69$, Pass $40 \leq P \leq 54$

Question 1

$$f(x,y) \equiv 4x^4 - 4x^3y - 7x^2y^2 + 4xy^3 + 3y^4 .$$

Express $f(x,y)$ as a product of 4 linear factors. **(8)**

Question 2

A hiker on a mountain walk has injured himself.

He rings the rescue station which is located at the point with coordinates $(2,1)$.

He reports that he is lying injured by a river bank where he can see a ruined tower, which his compass indicates that it is located South-West from his position.

It is known to the rescue station that the only river in the area has equation $x = 8$ and the ruined tower is located at the point with coordinates $(2,3)$ on the coordinate axes.

The rescuers set off immediately from the Rescue Station and travel directly towards the hiker. When the rescuers are half-way into their journey, the hiker rings again.

He says that he made a mistake in reading his compass and the ruined tower is in fact located North-West from his position.

The rescuers turn and head directly towards the true location of the hiker.

Calculate the angle, as a bearing, at which the rescuers are heading after the hikers second phone call.

Give the answer in the form $\mu\pi + \arctan\lambda$, where μ and λ are constants to be found.

(8)

Question 3

The straight lines l_1 and l_2 have the following vector equations

$$\mathbf{r}_1 = \mathbf{i} - 5\mathbf{j} + \lambda(4\mathbf{j} - \mathbf{k})$$

$$\mathbf{r}_2 = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k} + \mu(3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$

where λ and μ are scalar parameters.

a) Given that l_1 and l_2 intersect at some point Q, find the position vector of Q. **(3)**

b) Given further that the point P lies on l_1 and has position vector $\mathbf{i} + p\mathbf{j} - 3\mathbf{k}$, find the value of p. **(1)**

The point T lies on l_2 so that $\left|\overrightarrow{PQ}\right| = \left|\overrightarrow{QT}\right|$.

c) Determine the two possible position vectors for T. **(7)**

Question 4

The piecewise continuous function f is given below

$$f(x) \equiv \begin{cases} \sin x° & 0 \le x < 360 \\ \sin 2x° & 360 \le x < 720 \\ \sin 3x° & 720 \le x < 1080 \end{cases}$$

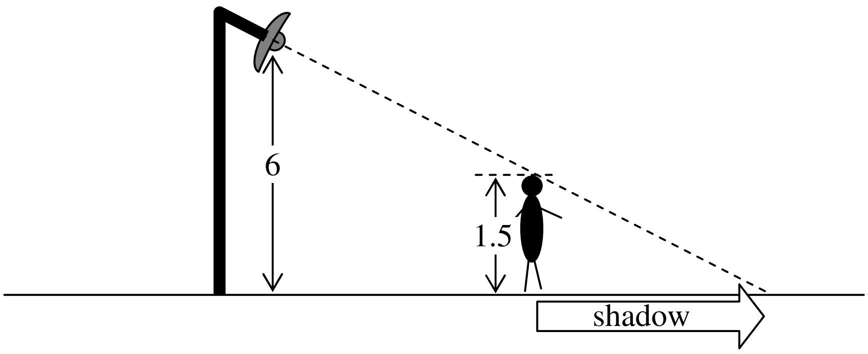
a) Sketch the graph of $f(x)$. **(2)**

b) Solve the equation …

i. … $f(x) = -\frac{1}{2}$. **(4)**

ii. … $f(x) = \cos x$. **(6)**

Question 5



The light bulb in a lamp-post stands 6 m high.

A boy, of height 1.5 m, is walking in a straight line away from the lamp-post at constant speed of 1.5 ms^{-1}.

Determine the rate at which the length of its shadow is increasing. **(7)**

Question 6

The circles C_1 and C_2 have respective equations

$$(x+1)^2 + (y+2)^2 = \frac{9}{4} \qquad \text{and} \qquad (x-5)^2 + (y-6)^2 = 36.$$

The point P lies on C_2 so that the distance of P from C_1 is least.

Determine the exact coordinates of P. **(7)**

Question 7

The quadratic equation

$$ax^2+bx+c=0,\ x\in\mathbb{R},$$

where a, b and c are constants, $a\neq 0$, has real roots which differ by 1.

Determine a simplified relationship between a, b and c. **(7)**

Question 8

The functions f and g are defined in the largest possible real domain and their equations are given in terms of a constant k by

$$f(x)=\frac{(3k^2+1)x-k+1}{x-k+3} \quad \text{and} \quad g(x)=\frac{7x+4k}{4x+10}.$$

Given that f and g are identical, determine the possible value or values of k. **(7)**

Question 9

The geometric mean of two positive numbers a and b is denoted by G.

The arithmetic mean of $\frac{1}{a}$ and $\frac{1}{b}$ is denoted by A.

Given further that the ratio $\frac{1}{A}:G=4:5$, determine the ratio between a and b. **(8)**

Question 10

Solve the cubic equation

$$4x^3-4\left(1+\sqrt{3}\right)x^2+\left(9+4\sqrt{3}\right)x-9=0,\ x\in\mathbb{R}.$$ **(8)**

Question 11

Find, in exact surd form, the only real solution of the following trigonometric equation

$$\arcsin(2x-1) - \arccos x = \frac{\pi}{6}.$$

The rejection of any additional solutions must be fully justified. **(13)**

Question 12

$$f(x) \equiv \frac{1}{x^{100}+100^{100}} \sum_{r=1}^{100} (x+r)^{100}, \quad x \in \mathbb{R},\ x \geq 0.$$

Use a formal method to find the equations of any asymptotes of $f(x)$. **(8)**

Question 13

By using an appropriate substitution or substitutions, show that

$$\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{4x\sin\left(x^2\right)}{\sin\left(x^2\right)+\sin\left(\ln 6 - x^2\right)}\, dx = \ln\left(\tfrac{3}{2}\right).$$ **(12)**

Question 14

The function f is defined as

$$f(x) = x^{-2x}, \quad x \in \mathbb{R}, \quad x > 0 .$$

Show that the value of $f''(x)$ at the stationary point of the function is

$$-2\mathrm{e}^{\frac{\mathrm{e}+2}{\mathrm{e}}} .$$

(12)

Question 15

It is given that

$$\sum_{r=1}^{n} u_r = 6^{n+1} - 10 \times 2^n + 4 ,$$

where u_n is the n^{th} term of a sequence.

Show clearly that

$$u_{n+2} = Au_{n+1} + Bu_n ,$$

where A and B are integers to be found. **(11)**

Question 16

Use integration by parts to find a simplified expression for

$$\int \left(1 + x - \frac{1}{x}\right) \mathrm{e}^{x+\frac{1}{x}} \, dx .$$

(10)

Question 17

The variable point P lies on the positive x axis and the variable point Q lies on the curve with equation

$$y = \frac{1}{x^2+4}, \quad x \in \mathbb{R}, \quad x \geq 0.$$

The x coordinate Q is always half the x coordinate of P.

The point P starts at the origin O and begins to move in the positive x direction at constant rate.

Determine the largest rate of area increase and the largest rate of area decrease of the triangle OPQ, as P is moving away from O. **(13)**

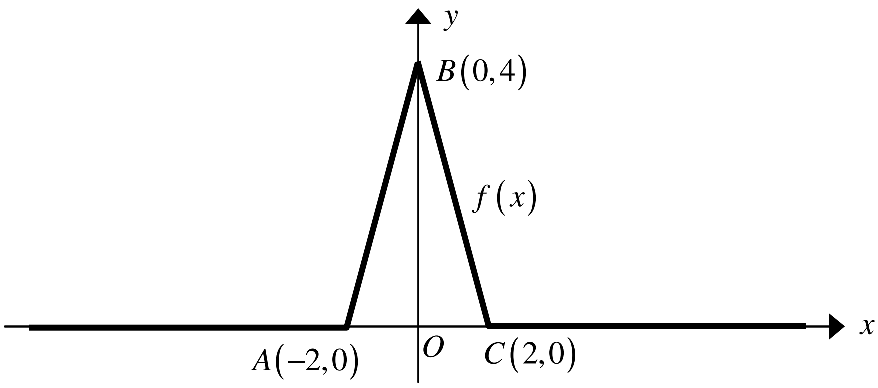
Question 18

Solve the trigonometric equation

$$8\cos 2x \cos 4x \cos 8x + 1 = 0, \quad 0 < x < \frac{\pi}{2},$$

giving the answers in terms of π. **(10)**

Question 19



The figure above shows the graph of the function $f(x)$, consisting entirely of straight line sections. The coordinates of the joints of these straight line sections which make up the graph of $f(x)$ are also marked in the figure.

Given further that

$$\int_{-2}^{2} k + f\left(x^2 - 4\right)\, dx = 0,$$

determine as an exact fraction the value of the constant k. **(12)**

Question 20

An unstable substance Z decomposes into two different substances X and Y, and at the same time X and Y recombine to reform substance Z. Two parts of Z decompose to one part of X and one part of Y, and at the same time one part of X and one part of Y recombine to reform two parts of Z. As a result at any given time the mass of X and Y are equal.

The rate at which the mass of Z reduces, due to decomposition, is k times the mass of Z present. The rate at which the mass of Z increases, due to reforming, is $4k$ times the product of the masses of X and Y.

Initially there are 6 grams of Z only.

Show that when the mass of Z is 5 grams, $kt = \frac{1}{5}\ln\left(\frac{8}{3}\right)$. **(16)**