IYGB - FP4 PAPER O - QUESTION 1

CONSTRUCTING THE TABLE

$*$	a	b	ab	e
a	e	ab	b	a
b	ab	e	a	b
ab	b	a	e	ab
e	a	b	ab	e

$a^2 = e$ GIVEN

$b^2 = e$ GIVEN

$ab = ba$ GIVEN

$a * ab = a^2 b = eb = b$

$b * ab = b * ba = b^2 a = ea = a$

$ab * a = ba * a = ba^2 = be = b$

$ab * b = ab^2 = ae = a$

$ab * ab = ba * ab = ba^2 b = beb = bb = b^2 = e$

IYGB - FP4 PAPER O - QUESTION 2

a) EVIDENTLY THE REQUIRED ANSWER IS

$$\underline{4} \times \underline{4} \times \underline{4} \times \underline{4} = 4^4 = 256$$

b) NO REPETITIONS IS A STANDARD PERMUTATION

$$^4P_4 = 4! = 24$$

c) A DIGIT CAN REPEATED AT MOST TWICE ...

ONE DOUBLE, 2 DISTINCT ... "2 PAIRS" ...

EASIER TO WORK THE COMPLEMENT

- "4 THE SAME" $= 4$
- 3 THE SAME – 1 DISTINCT $= ?$
- 2 PAIRS OR ONE DOUBLE & 2 DISTINCT $= ?$
- 4 DISTINCT $= 24$

TOTAL OF ALL $= 256$

3 THE SAME & ONE DIFFERENT – SAY 1, 1, 1, 2

THIS GIVES 4 ARRANGEMENTS

$\times 3$ (ONE WITH 2, ONE WITH 3, ONE WITH 4)

$\times 4$ (1,1,1 / 2,2,2 / 3,3,3 / 4,4,4)

48

THE REQUIRED NUMBER IS GIVEN BY

$$256 - 4 - 48 = 204$$

IYGB - FP4 PAPER O - QUESTION 3

a) SUPPOSE THAT THERE EXIST POSITIVE INTEGER a SUCH THAT

- ITS DIVISION BY 6 YIELDS A REMAINDER OF 4

$$a = 6n + 4, \quad n \in \mathbb{N}$$
$$2a = 12n + 8$$

- ITS DIVISION BY 12 GIVES REMAINDER 8

$$a = 12m + 8, \quad m \in \mathbb{N}$$

$$2a = 12n + 8$$
$$a = 12m + 8$$

$$a = 12n - 12m \quad \leftarrow \text{SUBTRACTING}$$
$$a = 12(n - m)$$

I.E a IS DIVISIBLE BY 12, WHICH IS A CONTRADICTION TO THE SECOND STATEMENT

$\therefore$ THERE IS NO SUCH INTEGER

b) SUPPOSE THAT THERE EXISTS A POSITIVE INTEGER a, SUCH THAT

- ITS DIVISION BY 6, YIELDS A REMAINDER OF 1, AND QUOTIENT q

$$\Rightarrow a = 6q + 1, \quad q \in \mathbb{N}$$
$$\Rightarrow a - 1 = 6q$$

- THE DIVISION OF a^2 BY q GIVES REMAINDER OF 1 AND QUOTIENT 984

$$\Rightarrow a^2 = q \times 984 + 1, \quad q \in \mathbb{N}$$
$$\Rightarrow a^2 - 1 = 984q$$
$$\Rightarrow (a+1)(a-1) = 984q$$
$$\Rightarrow (a+1) \times 6q = 984q$$

IYGB - FP4 PAPER O - QUESTION 3

$$\Longrightarrow \quad 6\cancel{a}(a+1) = 984\cancel{a} \qquad a \neq 0$$

THIS CAN BE SATISFIED IF 984 IS DIVISIBLE BY 6

WHICH IT IS, AS $984 \div 6 = 164$

$\therefore$ THERE EXISTS SUCH POSITIVE INTEGER a TO FIND IT

SIMPLY $a+1 = 164$ IE $a = 163$

IYGB - FP4 PAPER O - QUESTION 4

SET UP A REDUCTION FORMULA IN TERMS OF n, $n \in \mathbb{N}$

$$I_n = \int_0^1 x^n e^{-x^2}\,dx$$

$$I_n = \int_0^1 x^{n-1}\left(xe^{-x^2}\right)dx$$

PROCEED BY INTEGRATION BY PARTS

x^{n-1}	$(n-1)x^{n-2}$
$-\frac{1}{2}e^{-x^2}$	xe^{-x^2}

$$I_n = \left[-\tfrac{1}{2}x^{n-1}e^{-x^2}\right]_0^1 - \int_0^1 -\tfrac{1}{2}(n-1)x^{n-2}e^{-x^2}\,dx$$

$$I_n = -\tfrac{1}{2}e^{-1} + \tfrac{1}{2}(n-1)\int_0^1 x^{n-2}e^{-x^2}\,dx$$

$$I_n = -\tfrac{1}{2}e^{-1} + \tfrac{1}{2}(n-1)I_{n-2}$$

USING THE FORMULA DERIVED TO OBTAIN I_5

$$\Rightarrow I_5 = -\tfrac{1}{2}e^{-1} + \tfrac{1}{2}\times 4\times I_3 = -\tfrac{1}{2}e^{-1} + 2I_3$$

$$\Rightarrow I_5 = -\tfrac{1}{2}e^{-1} + 2\left[-\tfrac{1}{2}e^{-1} + \tfrac{1}{2}\times 2\,I_1\right] = -\tfrac{3}{2}e^{-1} + 2I_1$$

$$\Rightarrow I_5 = -\tfrac{3}{2}e^{-1} + 2\int_0^1 xe^{-x^2}\,dx$$

BY RECOGNITION

$$\Rightarrow I_5 = -\tfrac{3}{2}e^{-1} + 2\left[-\tfrac{1}{2}e^{-x^2}\right]_0^1$$

$$\Rightarrow I_5 = -\tfrac{3}{2}e^{-1} + 2\left[-\tfrac{1}{2}e^{-1} + \tfrac{1}{2}\right] = -\tfrac{5}{2}e^{-1} + 1$$

$$\Rightarrow I_5 = 1 - \frac{5}{2e}$$

$$\Rightarrow I_5 = \frac{2e-5}{2e}$$

AS REQUIRED

IYGB – FP4 PAPER O – QUESTION 5

a) USING THE FACT THAT $(\lambda-2)$ IS A FACTOR

$$\Rightarrow 2\lambda^3 - 7\lambda^2 + \lambda + 10 = 0$$
$$\Rightarrow 2\lambda^2(\lambda-2) - 3\lambda(\lambda-2) - 5(\lambda-2) = 0$$
$$\Rightarrow (\lambda-2)(2\lambda^2 - 3\lambda - 5) = 0$$
$$\Rightarrow (\lambda-2)(2\lambda-5)(\lambda+1) = 0$$

$$\lambda = \begin{cases} 2 \\ 5/2 \\ -1 \end{cases}$$

b) BY C–H THEOREM, A MATRIX MUST SATISFY ITS CHARACTERISTIC EQUATION

$$\Rightarrow 2\underline{A}^3 - 7\underline{A}^2 + \underline{A} + 10\underline{I} = \underline{0}$$
$$\Rightarrow 2\underline{A}^3\underline{A} - 7\underline{A}^2\underline{A} + \underline{A}\,\underline{A} + 10\underline{I}\,\underline{A} = \underline{0}$$
$$\Rightarrow 2\underline{A}^4 - 7\underline{A}^3 + \underline{A}^2 + 10\underline{A} = 0$$
$$\Rightarrow 2\underline{A}^4 - 7\underline{A}^3 + \underline{A}^2 + 10\left[-2\underline{A}^3 + 7\underline{A}^2 - 10\underline{I}\right] = 0$$
$$\Rightarrow 2\underline{A}^4 - 7\underline{A}^3 + \underline{A}^2 - 20\underline{A}^3 + 70\underline{A}^2 - 100\underline{I} = \underline{0}$$
$$\Rightarrow 2\underline{A}^4 - 27\underline{A}^3 + 71\underline{A}^2 - 100\underline{I} = \underline{0}$$
$$\Rightarrow 2\underline{A}^4 + 71\underline{A}^2 = 27\underline{A}^3 + 100\underline{I}$$

c) I) $\underline{A}\underline{u} = \lambda\underline{u} = 2\underline{u} = \begin{pmatrix} 4 \\ -8 \\ -10 \end{pmatrix}$

II) $\underline{A}^2\underline{v} = \underline{A}^2\left(\frac{1}{5}\underline{u}\right) = \frac{1}{5}\underline{A}(\underline{A}\underline{u})$

$$= \frac{1}{5}\underline{A}(2\underline{u}) = \frac{2}{5}\underline{A}\underline{u}$$
$$= \frac{2}{5}\begin{pmatrix} 4 \\ -8 \\ -10 \end{pmatrix} = \begin{pmatrix} 1 \\ -3.2 \\ -4 \end{pmatrix}$$

d)
$$\Rightarrow \underline{A}\underline{x} = \underline{v}$$
$$\Rightarrow \underline{A}\underline{x} = \frac{1}{5}\underline{u}$$
$$\Rightarrow 10\underline{A}\underline{x} = 2\underline{u}$$
$$\Rightarrow \underline{A}(10\underline{x}) = 2\underline{u}$$

$$\Rightarrow 10\underline{x} = \underline{u}$$
$$\Rightarrow \underline{x} = \frac{1}{10}\underline{u}$$
$$\Rightarrow \underline{x} = \begin{pmatrix} 0.2 \\ -0.4 \\ -0.5 \end{pmatrix}$$

IYGB - FP4 PAPER O - QUESTION 6

USING THE STANDARD ARCLENGTH FORMULA IN CARTESIAN

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$y = \ln(1-x^2)$$

$$\frac{dy}{dx} = \frac{1}{1-x^2} \times (-2x)$$

$$s = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{1 + \left(\frac{-2x}{1-x^2}\right)^2}\, dx$$

$$s = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{1 + \frac{4x^2}{1-2x^2+x^4}}\, dx$$

$$s = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{\frac{1-2x^2+x^4+4x^2}{1-2x^2+x^4}}\, dx$$

$$s = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{\frac{x^4+2x^2+1}{x^4-2x^2+1}}\, dx$$

$$s = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{\frac{(x^2+1)^2}{(x^2-1)^2}}\, dx$$

$$s = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{x^2+1}{x^2-1}\right| dx$$

$$s = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1+x^2}{1-x^2}\, dx \quad // \text{ AS REQUIRED}$$

TO INTEGRATE THE EXPRESSION, NOTE THAT THE INTEGRAND IS EVEN AND THE DOMAIN IS SYMMETRICAL

$$s = 2\int_{0}^{\frac{1}{2}} \frac{1+x^2}{1-x^2}\, dx$$

IYGB - FP4 PAPER D - QUESTION 6

MANIPULATE THE IMPROPER FRACTION IN THE INTEGRAND

$$S = 2\int_0^{\frac{1}{2}} \frac{2-(1-x^2)}{(1-x^2)}\,dx = 2\int_0^{\frac{1}{2}} \frac{2}{1-x^2} - 1\,dx.$$

$$S = \int_0^{\frac{1}{2}} \frac{4}{(1-x)(1+x)} - 2\,dx$$

PARTIAL FRACTIONS BY INSPECTION

$$S = \int_0^{\frac{1}{2}} \frac{2}{1-x} + \frac{2}{1+x} - 2\,dx$$

$$S = \Big[-2\ln|1-x| + 2\ln|1+x| - 2x\Big]_0^{\frac{1}{2}}$$

$$S = \left(-2\ln\tfrac{1}{2} + 2\ln\tfrac{3}{2} - 1\right) - \left(\cancel{-2\ln 1} + \cancel{2\ln 1} - 0\right)$$

$$S = 2\ln\tfrac{3}{2} - \ln\tfrac{1}{2} - 1$$

$$S = 2\left[\ln\tfrac{3}{2} - \ln\tfrac{1}{2}\right] - 1$$

$$S = 2\left[\ln\tfrac{3}{2} + \ln 2\right] - 1$$

$$S = 2\ln 3 - 1$$

IYGB - FP4 PAPER O - QUESTION 7

WORK AS FOLLOWS

$$w = \frac{1}{z} \implies z = \frac{1}{w}$$

$$\implies z + \tfrac{1}{2}i = \frac{1}{w} + \frac{1}{2}i$$

$$\implies z + \tfrac{1}{2}i = \frac{2 + wi}{2w}$$

TAKING MODULI ON BOTH SIDES

$$\implies \left| z + \tfrac{1}{2}i \right| = \left| \frac{2 + wi}{2w} \right|$$

$$\implies \frac{1}{2} = \frac{|2 + wi|}{|2w|}$$

$$\implies |w| = |2 + wi|$$

LET $w = u + iv$

$$\implies |u + iv| = |2 + i(u + iv)|$$

$$\implies |u + iv| = |2 + ui - v|$$

$$\implies |u + iv| = |(2 - v) + iu|$$

$$\implies \sqrt{u^2 + v^2} = \sqrt{(2 - v)^2 + u^2}$$

$$\implies \cancel{u^2} + \cancel{v^2} = 4 - 4v + \cancel{v^2} + \cancel{u^2}$$

$$\implies 4v = 4$$

$$\implies v = 1$$

[OR $y = 1$]