IYGB - FP3 PAPER V - QUESTION 1

METHOD A

$$\frac{x-7}{x} \leqslant \frac{5}{x(x-3)}$$

$$\Rightarrow \frac{x-7}{x} - \frac{5}{x(x-3)} \leqslant 0$$

$$\Rightarrow \frac{(x-7)(x-3)-5}{x(x-3)} \leqslant 0$$

$$\Rightarrow \frac{x^2-10x+21-5}{x(x-3)} \leqslant 0$$

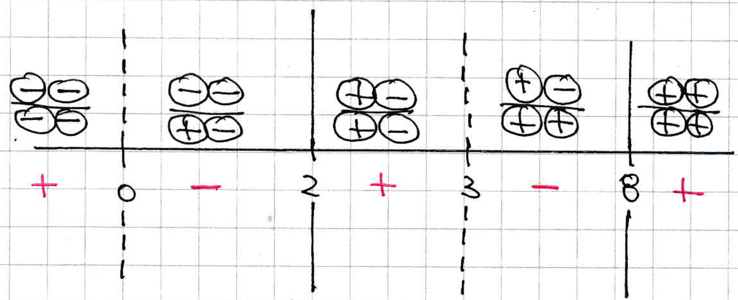
$$\Rightarrow \frac{x^2-10x+16}{x(x-3)} \leqslant 0$$

$$\Rightarrow \frac{(x-2)(x-8)}{x(x-3)} \leqslant 0$$

CRITICAL VALUES

$x =$ 2, 8 NUMERATOR ZERO

$x =$ 0, 3 VERTICAL ASYMPTOTES (DENOMINATOR ZERO)



LOOKING FOR MINUS OVERALL

$$0 < x \leqslant 2 \quad \cup \quad 3 < x \leqslant 8$$

METHOD B

$$\frac{x-7}{x} \leqslant \frac{5}{x(x-3)}$$

$$\Rightarrow \frac{(x-7)x}{x^2} \leqslant \frac{5x(x-3)}{x^2(x-3)^2}$$

$$\Rightarrow (x-7)x^3(x-3)^2 \leqslant 5x^3(x-3)$$

$$\Rightarrow x^3(x-3)^2(x-7) - 5x^3(x-3) \leqslant 0$$

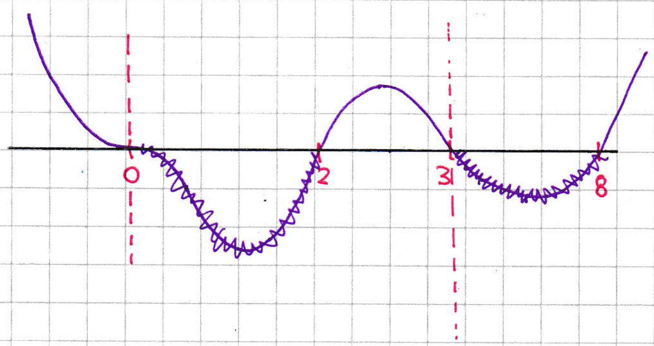
$$\Rightarrow x^3(x-3)\left[(x-3)(x-7)-5\right] \leqslant 0$$

$$\Rightarrow x^3(x-3)(x^2-10x+21-5) \leqslant 0$$

$$\Rightarrow x^3(x-3)(x^2-10x+16) \leqslant 0$$

$$\Rightarrow x^3(x-3)(x-2)(x-8) \leqslant 0$$

SKETCH THE LHS – MARK ORIGINAL ASYMPTOTES



$$0 < x \leqslant 2 \quad \cup \quad 3 < x \leqslant 8$$

IYGB - FP3 PAPER V - QUESTION 2

a) FORM A STANDARD TABLE FOR SIMPSON RULE IN TERMS OF k

x	1	1.5	2	2.5	3
$\frac{k}{2x-1}$	k	$\frac{k}{2}$	$\frac{k}{3}$	$\frac{k}{4}$	$\frac{k}{5}$.
	FIRST	ODD	EVEN	ODD	LAST

$$\int_1^3 \frac{k}{2x-1}\,dx \simeq \frac{\text{THICKNESS}}{3}\left[\text{FIRST} + \text{LAST} + 4\times\text{ODD} + 2\times\text{EVEN}\right]$$

$$1.46 \simeq \frac{0.5}{3}\left[k + \frac{k}{5} + 4\left(\frac{k}{2}+\frac{k}{4}\right) + 2\times\frac{k}{3}\right]$$

$$1.46 \simeq \frac{1}{6}\left[\frac{73}{15}k\right]$$

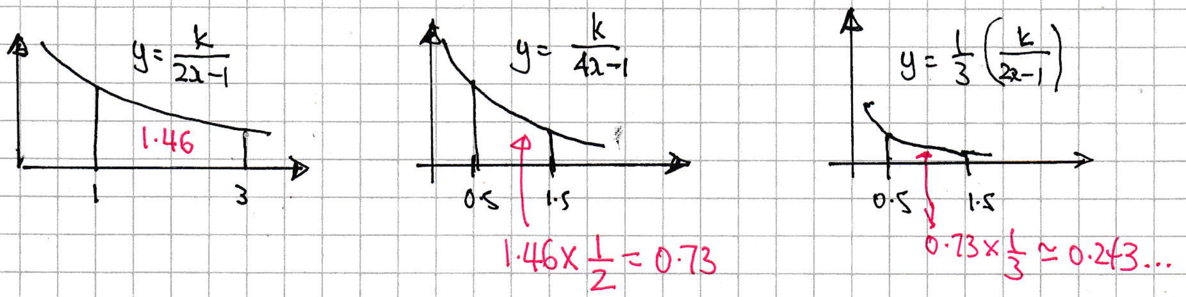
$$1.46 \simeq \frac{73}{90}k.$$

$$k \simeq \frac{9}{5}$$

b) EXAMINING THE NEW INTEGRAL

$$\int_{0.5}^{1.5} \frac{k}{12x-3}\,dx = \frac{1}{3}\int_{0.5}^{1.5}\frac{k}{4x-1}\,dx = \frac{1}{3}\int_{0.5}^{1.5}\frac{k}{2(2x)-1}\,dx$$

IN TERMS OF TRANSFORMATIONS



$$\therefore \int_{0.5}^{1.5}\frac{k}{12x-3}\,dx \simeq 0.24$$

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IYGB − FP3 PAPER V − QUESTION 3

THE LIMIT IS OF THE TYPE (ZERO) × (− INFINITY) SO IT CAN BE MANIPULATED TO USE L'HOSPITAL RULE

$$\lim_{x\to0^+}\left[x^p \ln x\right] = \lim_{x\to0^+}\left[\frac{\ln x}{\frac{1}{x^p}}\right] \quad \leftarrow \text{TYPE } \text{"}\frac{-\infty}{\infty}\text{"}$$

DIFFERENTIATING "TOP" & BOTTOM W.R.T x

$$= \lim_{x\to0^+}\left[\frac{\ln x}{x^{-p}}\right] = \lim_{x\to0^+}\left[\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x^{-p})}\right] = \lim_{x\to0^+}\left[\frac{\frac{1}{x}}{-px^{-p-1}}\right]$$

$$= -\frac{1}{p}\lim_{x\to0^+}\left[\frac{\frac{1}{x}}{\frac{1}{x^{p+1}}}\right] = -\frac{1}{p}\lim_{x\to0^+}\left[\frac{x^{p+1}}{x}\right] = -\frac{1}{p}\lim_{x\to0^+}\left[x^p\right]$$

$$= 0$$

IYGB - FP3 PAPER V - QUESTION 4

a) STANDARD ELIMINATIONS BY SUBSTITUTIONS

(I) $y = 9 - 5x - 6z$

SUBSTITUTE INTO THE OTHER TWO EQUATIONS

$$\left\{\begin{array}{l} 3x + 6(9-5x-6z) + 2z = 8 \\ 4x + 2(9-5x-6z) - 9z = 75 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} 3x + 54 - 30x - 36z + 2z = 8 \\ 4x + 18 - 10x - 12z - 9z = 75 \end{array}\right\}$$

$$\left\{\begin{array}{l} -27x - 34z = -46 \\ -6x - 21z = 57 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} 27x + 34z = 46 \\ 2x + 7z = -19 \end{array}\right\} \begin{array}{l} \times 2 \\ \times 27 \end{array}$$

$$\left\{\begin{array}{l} 54x + 68z = 92 \\ 54x + 189z = -513 \end{array}\right\} \Rightarrow 121z = -605$$

$$\therefore z = -5$$ AS REQUIRED

FINALLY WE HAVE

$2x + 7z = -19$
$2x - 35 = -19$
$2x = 16$
$x = 8$

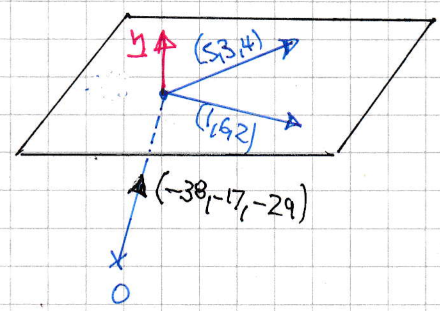
$y = 9 - 5x - 6z$
$y = 9 - 40 + 30$
$y = -1$

b) THE DIRECTION OF THE LINE IS $(-6,-2,9)$ SCALED TO $(6,2,-9)$

FOR THE PLANE WE HAVE:

$$\underline{n} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 5 & 3 & 4 \\ 1 & 6 & 2 \end{vmatrix} = (-18,-6,27)$$

SCALED TO $(6,2,-9)$



AS $\underline{n}$ IS PARALLEL TO THE LINE DIRECTION, L IS PERPENDICULAR TO Π

IYGB - FP3 PAPER V - QUESTION 4

c) WORKING IN PARAMETRIC FOR THE LINE & IN CARTESIAN FOR THE PLANE

$$\Pi: \ 6x+2y-9z = \text{CONSTANT}$$

$$\therefore 6(-38)+2(-17)-9(-29) = \text{CONSTANT}$$

$$\therefore \text{CONSTANT} = -228+261-34$$

$$\therefore \text{CONSTANT} = -1$$

$$L: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -29-6t \\ -9-2t \\ 46+9t \end{pmatrix}$$

$$\Rightarrow 6x+2y-9z=-1$$

$$\Rightarrow 6(-29-6t)+2(-9-2t)-9(46+9t)=-1$$

$$\Rightarrow -174-36t-18-4t-414-81t=-1$$

$$\Rightarrow -121t = 605$$

$$\Rightarrow t=-5$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

ALTERNATIVE TO PART (c) USING PART (a) AND IN FULL PARAMETRIC

$$\underline{r}_1 = \underline{r}_2 \Rightarrow \begin{pmatrix} -29-6t \\ -9-2t \\ 46+9t \end{pmatrix} = \begin{pmatrix} -38+5\lambda+\mu \\ -17+3\lambda+6\mu \\ -29+4\lambda+2\mu \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -5\lambda-\mu-6t \\ -3\lambda-6\mu-2t \\ -4\lambda-2\mu+9t \end{pmatrix} = \begin{pmatrix} -9 \\ -8 \\ -75 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 5\lambda+\mu+6t \\ 3\lambda+6\mu+2t \\ 4\lambda+2\mu-9t \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \\ 75 \end{pmatrix}$$

$\lambda \mapsto x = 8$
$\mu \mapsto y = -1$
$t \mapsto z = -5$

USING $t=-5$ WE OBTAIN AS BEFORE $(1,1,1)$

IYGB – FP3 PAPER V – QUESTION 5

USE THE SUBSTITUTION $v = \sqrt{y+1}$

$$v^2 = y+1$$
$$2v\frac{dv}{dx} = \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} + \sqrt{y+1} = y+1$$

$$\Rightarrow 2v\frac{dv}{dx} + v = v^2$$

$v \neq 0$

$$\Rightarrow 2\frac{dv}{dx} + 1 = v$$

$$\Rightarrow 2\frac{dv}{dx} = v-1$$

$$\Rightarrow \frac{2}{v-1}\,dv = 1\,dx$$

INTEGRATE BOTH SIDES

$$\Rightarrow 2\ln|v-1| = x + C$$

$$\Rightarrow \ln|v-1| = \frac{1}{2}x + D$$

$$\Rightarrow |v-1| = e^{\frac{1}{2}x+D}$$

$$\Rightarrow |v-1| = Ae^{\frac{1}{2}x}$$

$$\Rightarrow |+\sqrt{y+1}-1| = Ae^{\frac{1}{2}x}$$ (ORIGINAL SUBSTITUTION)

APPLY CONDITION

$y(0)=3 \Rightarrow |2-1| = A$

$\Rightarrow A=1$

$$\Rightarrow |\sqrt{y+1}-1| = e^{\frac{1}{2}x}$$

IYGB - FP3 PAPER V - QUESTION 5

GETTING THE RESULTS AND TIDY UP

$$\left|\sqrt{y+1}-1\right| = e^{\frac{1}{2}x}$$

$$\sqrt{y+1}-1 = e^{\frac{1}{2}x}$$
$$\sqrt{y+1} = e^{\frac{1}{2}x}+1$$
$$\left(\sqrt{y+1}\right)^2 = e^{x}+2e^{\frac{1}{2}x}+1$$
$$y+1 = e^{x}+2e^{\frac{1}{2}x}+1$$
$$y = e^{x}+2e^{\frac{1}{2}x}$$

$$-\sqrt{y+1}+1 = e^{\frac{1}{2}x}$$
$$\sqrt{y+1}-1 = -e^{\frac{1}{2}x}$$
$$\sqrt{y+1} = 1-e^{\frac{1}{2}x}$$
$$y+1 = 1-2e^{\frac{1}{2}x}+e^{x}$$
$$y = e^{x}-2e^{\frac{1}{2}x}$$

$$\therefore\ y = e^{x} \pm 2e^{\frac{1}{2}x}$$

IYGB - FP3 PAPER V - QUESTION 6

$$\boxed{\frac{dy}{dx} = \frac{e^{x+y}}{3x+y+k} \qquad x=0,\ y=0 \quad \& \quad x=0.1 \quad y=0.025}$$

USING $\left(\frac{dy}{dx}\right)_r \approx \frac{y_{r+1}-y_r}{h}$

$$\Longrightarrow \quad y'_r \approx \frac{y_{r+1}-y_r}{h}$$

$$\Longrightarrow \quad y_{r+1} \approx hy'_r + y_r$$

$$\Longrightarrow \quad \boxed{y_{r+1} \approx h\left(\frac{e^{x_r+y_r}}{3x_r+y_r+k}\right)+y_r}$$

$$\Longrightarrow \quad y_1 = h\left(\frac{e^{x_0+y_0}}{3x_0+y_0+k}\right)+y_0$$

$$\Longrightarrow \quad 0.025 = 0.1\left(\frac{e^{0+0}}{0+0+k}\right)+0$$

$$\Longrightarrow \quad 0.025k = 0.1$$

$$\Longrightarrow \quad k = 4$$

$x_0 = 0$
$y_0 = 0$
$y_1 = 0.025$
$h = 0.1$

REAPPLYING THE FORMULA ONCE MORE

$$\Longrightarrow \quad y_2 = h\left(\frac{e^{x_1+y_1}}{3x_1+y_1+k}\right)+y_1$$

$$\Longrightarrow \quad y_2 = 0.1\left(\frac{e^{0.1+0.025}}{3\times0.1+0.025+4}\right)+0.025$$

$$\Longrightarrow \quad y_2 \approx 0.0512$$

3 s.f.

IYGB – FP3 PAPER V – QUESTION 7

USING THE SUBSTITUTION GIVEN

- $t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)$

$\frac{dt}{dx} = \frac{1}{2}(1+t^2)$

$\frac{dx}{dt} = \frac{2}{1+t^2}$

$dx = \frac{2}{1+t^2}\,dt$

- $\cos x = 2\cos^2\frac{x}{2} - 1$

$= \frac{2}{\sec^2\frac{x}{2}} - 1$

$= \frac{2}{1+\tan^2\frac{x}{2}} - 1$

$= \frac{2}{1+t^2} - 1$

$= \frac{2-(1+t^2)}{1+t^2}$

$= \frac{1-t^2}{1+t^2}$

- $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

$\sin x = 2\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\cos^2\frac{x}{2}$

$\sin x = 2\tan\frac{x}{2}\cos^2\frac{x}{2}$

$\sin x = 2t\left(\frac{1}{\sec^2\frac{x}{2}}\right)$

$\sin x = 2t\left(\frac{1}{1+\tan^2\frac{x}{2}}\right)$

$\sin x = \frac{2t}{1+t^2}$

NOW TRANSFORM THE INTEGRAL NOTING THE CHANGE OF LIMITS

$$\int_0^{\frac{\pi}{2}} \frac{1}{5+3\sin x+4\cos x}\,dx = \int_0^1 \frac{1}{5+3\left(\frac{2t}{1+t^2}\right)+4\left(\frac{1-t^2}{1+t^2}\right)}\left(\frac{2}{1+t^2}\,dt\right)$$

$$= \int_0^1 \frac{1}{5+\frac{6t}{1+t^2}+\frac{4-4t^2}{1+t^2}}\left(\frac{2}{1+t^2}\right)dt = \int_0^1 \frac{2}{5(1+t^2)+6t+4-4t^2}\,dt$$

$$= \int_0^1 \frac{2}{t^2+6t+9}\,dt = \int_0^1 \frac{2}{(t+3)^2}\,dt$$

$$= \left[-\frac{2}{t+3}\right]_0^1 = \left[\frac{2}{t+3}\right]_1^0 = \frac{2}{3}-\frac{1}{2} = \frac{1}{6}$$

IYGB – FP3 PAPER V – QUESTION 8

- START BY FINDING THE EQUATION OF THE NORMAL AT $P\left(\frac{p}{2}, \frac{1}{2p}\right)$

$$\Rightarrow 4xy = 1$$

$$\Rightarrow y = \frac{1}{4x} \quad \left(\text{OR } x = \frac{1}{4y}\right)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{4x^2}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=\frac{p}{2}} = -\frac{1}{4\left(\frac{1}{4}p^2\right)} = -\frac{1}{p^2}.$$

- THE EQUATION OF THE NORMAL IS GIVEN BY

$$y - \frac{1}{2p} = p^2\left(x - \frac{1}{2}p\right)$$

- SOLVING SIMULTANEOUSLY WITH $4xy = 1$

$$\Rightarrow \frac{1}{4x} - \frac{1}{2p} = p^2x - \frac{1}{2}p^3 \qquad \times 4px$$

$$\Rightarrow p - 2x = 4p^3x^2 - 2p^4x$$

$$\Rightarrow 0 = 4p^3x^2 + (2 - 2p^4)x - p$$

$$\frac{\alpha + \beta}{2} = \frac{1}{2}\left(-\frac{b}{a}\right) = \frac{1}{2}\left[\frac{2p^4 - 2}{4p^3}\right] = \boxed{\frac{p^4 - 1}{4p^3}}$$

- REPEAT THE PROCESS FOR y

$$\Rightarrow y - \frac{1}{2p} = p^2\left(\frac{1}{4y} - \frac{1}{2}p\right)$$

$$\Rightarrow y - \frac{1}{2p} = \frac{p^2}{4y} - \frac{p^3}{2} \qquad \times 4py$$

$$\Rightarrow 4py^2 - 2y = p^3 - 2p^4y$$

IYGB - FP3 PAPER V - QUESTION 8

$$\Longrightarrow 4py^2 + (2p^4-2) - p^3 = 0$$

$$\frac{\alpha+\beta}{2} = \frac{1}{2}\left(-\frac{b}{a}\right) = \frac{1}{2}\left(\frac{2-2p^4}{4p}\right) = \boxed{\frac{1-p^4}{4p}}$$

$\therefore$ THE COORDINATES OF THE MIDPOINT OF PQ ARE

$$\left(\frac{p^4-1}{4p^3}, \frac{1-p^4}{4p}\right)$$

ELIMINATING THE PARAMETER p

- $x = \dfrac{p^4-1}{4p^3} = \dfrac{1}{p^2}\left[\dfrac{p^4-1}{4p}\right]$
- $y = -\dfrac{p^4-1}{4p}$

$$\frac{1}{y} = -\frac{4p}{p^4-1}$$

$$\frac{x}{y} = \frac{1}{p^2}\left(\frac{\cancel{p^4-1}}{\cancel{4p}}\right)\left(-\frac{\cancel{4p}}{\cancel{p^4-1}}\right)$$

$$p^2 = -\frac{y}{x}$$

FINALLY WE OBTAIN

$$y^2 = \frac{(p^4-1)^2}{16p^2} = \frac{\left[\left(-\frac{y}{x}\right)^2-1\right]^2}{16\left(-\frac{y}{x}\right)} = \frac{\left(\frac{y^2}{x^2}-1\right)^2}{-\frac{16y}{x}} = \frac{\frac{(y^2-x^2)^2}{x^4}}{-\frac{16y}{x}}$$

$$\Longrightarrow y^2 = \frac{x(y^2-x^2)^2}{-16yx^4}$$

$$\Longrightarrow -16x^4y^3 = x(y^2-x^2)^2$$

$$\Longrightarrow -16x^3y^3 = (y^2-x^2)^2$$

$$\Longrightarrow (y^2-x^2)^2 + 16x^3y^3 = 0$$

AS REQUIRED