IYGB – FP2 PAPER R – QUESTION 1

LOOK FOR AN INTEGRATING FACTOR

$$IF = e^{\int \tan x \, dx} = e^{\ln|\sec x|} = \sec x = \frac{1}{\cos x}$$

MULTIPLY THROUGH BY THE INTEGRATING FACTOR TO MAKE EXACT

$$\Rightarrow \frac{1}{\cos x}\frac{dy}{dx} + y\tan x \frac{1}{\cos x} = e^{2x}\cos x \frac{1}{\cos x}$$

$$\Rightarrow \sec x \frac{dy}{dx} + y \tan x \sec x = e^{2x}.$$

$$\Rightarrow \frac{d}{dx}\left(y\sec x\right) = e^{2x}$$

NOTE THAT
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\Rightarrow y\sec x = \int e^{2x}\,dx$$

$$\Rightarrow y\sec x = \frac{1}{2}e^{2x} + C$$

$$\Rightarrow \frac{y}{\cos x} = \frac{1}{2}e^{2x} + C$$

$$\Rightarrow y = \frac{1}{2}e^{2x}\cos x + C\cos x$$

APPLY CONDITION $y(0) = 2$

$$\Rightarrow 2 = \frac{1}{2}\times 1 \times 1 + C \times 1$$

$$\Rightarrow 2 = \frac{1}{2} + C$$

$$\Rightarrow C = \frac{3}{2}$$

FINALLY WE OBTAIN

$$\Rightarrow y = \frac{1}{2}e^{2x}\cos x + \frac{3}{2}\cos x$$

$$\Rightarrow y = \frac{1}{2}\left(e^{2x}+3\right)\cos x$$

AS REQUIRED

IYGB - FP2 PAPER Q - QUESTION 2

a) DIFFERENTIATE & MANIPULATE BY THE PRODUCT RULE

- $y = (1+x)^2 \cos x$
- $\frac{dy}{dx} = 2(1+x)\cos x - (1+x)^2 \sin x$
- $\frac{d^2y}{dx^2} = 2\cos x - 2(1+x)\sin x - 2(1+x)\sin x - (1+x)^2 \cos x$

$= \left[2 - (1+x)^2\right] \cos x - 4(1+x)\sin x$

$= (2 - 1 - 2x - x^2)\cos x - 4(1+x)\sin x$

$= (1 - 2x - x^2)\cos x - 4(1+x)\sin x$

- $\frac{d^3y}{dx^3} = (-2-2x)\cos x - (1-2x-x^2)\sin x - 4\sin x - 4(1+x)\cos x$

$\frac{d^3y}{dx^3} = (-2-2x-4-4x)\cos x + (-1+2x+x^2-4)\sin x$

$\frac{d^3y}{dx^3} = (-6x-6)\cos x + (x^2+2x-5)\sin x$

$\frac{d^3y}{dx^3} = (x^2+2x-5)\sin x - 6(x+1)\cos x$ // AS REQUIRED

b) OBTAIN ALL THE DERIVATIVES AT $x=0$

$y\big|_{x=0} = 1 \qquad \frac{dy}{dx}\Big|_{x=0} = 2$

$\frac{d^2y}{dx^2}\Big|_{x=0} = 1$

$\frac{d^3y}{dx^3}\Big|_{x=0} = -6$

IYGB - FP2 PAPER R - QUESTION 2

BY THE MACLAURIN THEOREM

$$y = y_0 + xy_0' + \frac{x^2}{2!}y_0'' + \frac{x^3}{3!}y_0''' + O(x^4)$$

$$(1+x)^2\cos x = 1 + x \times 2 + \frac{x^2}{2} \times 1 + \frac{x^3}{6} \times (-6) + O(x^4)$$

$$(1+x)^2\cos x = 1 + 2x + \frac{1}{2}x^2 - x^3 + O(x^4)$$

IYGB – FP2 PAPER R – QUESTION 3

AUXILIARY EQUATION FOR THE O.D.E IS

$$\lambda^2+\lambda-2=0$$
$$(\lambda-1)(\lambda+2)=0$$
$$\lambda = \begin{cases} 1 \\ -2 \end{cases}$$

∴ COMPLEMENTARY FUNCTION

$$y = Ae^{x} + Be^{-2x}$$

AS THE R.H.S CONTAINS $6e^{-2x}$ WHICH IS PART OF THE SOLUTION FOR THE PARTICULAR INTEGRAL WE TRY

$$y = Pxe^{-2x}$$

$$\frac{dy}{dx} = Pe^{-2x} - 2Pxe^{-2x}$$

$$\frac{d^2y}{dx^2} = -2Pe^{-2x} - 2Pe^{-2x} + 4Pxe^{-2x} = 4Pxe^{-2x} - 4Pe^{-2x}$$

SUB INTO THE O.D.E

$$\left(\cancel{4Pxe^{-2x}} - 4Pe^{-2x}\right) + \left(Pe^{-2x} - \cancel{2Pxe^{-2x}}\right) - \cancel{2}\left(\cancel{Pxe^{-2x}}\right) \equiv 6e^{-2x}$$

$$-3Pe^{-2x} \equiv 6e^{-2x}$$

$$P = -2$$

∴ PARTICULAR INTEGRAL IS

$$y = -2xe^{-2x}$$

∴ GENERAL SOLUTION IS

$$y = Ae^{x} + Be^{-2x} - 2xe^{-2x}$$

IYGB - FP2 PAPER R - QUESTION 3

DIFFERENTIATE AND APPLY CONDITIONS

$$y = Ae^{x} + Be^{-2x} - 2xe^{-2x}$$

$$\frac{dy}{dx} = Ae^{x} - 2Be^{-2x} - 2e^{-2x} + 4xe^{-2x}$$

- $x=0 \quad y=3 \implies 3 = A+B$
- $x=0 \quad \frac{dy}{dx}=-2 \implies -2 = A-2B-2$

$$0 = A-2B$$

$$A = 2B$$

$$\therefore \quad 3 = 2B + B$$

$$B=1 \quad \& \quad A=2$$

FINALLY WE HAVE

$$y = 2e^{x} + e^{-2x} - 2xe^{-2x}$$

IYGB - FP2 PAPER R - QUESTION 4

USING PARTIAL FRACTIONS

$$\frac{1}{k(k+1)(k+2)} \equiv \frac{\frac{1}{1\times 2}}{k} - \frac{\frac{1}{-1\times 1}}{k+1} + \frac{\frac{1}{-2(-1)}}{k+2}$$

$$\frac{1}{k(k+1)(k+2)} \equiv \frac{\frac{1}{2}}{k} - \frac{1}{k+1} + \frac{\frac{1}{2}}{k+2}$$

DOUBLING THE ABOVE IDENTITY FOR SIMPLICITY

$$\boxed{\frac{2}{k(k+1)(k+2)} \equiv \frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2}}$$

- $k=1$ $\quad \frac{2}{1\times 2\times 3} = \frac{1}{1} - \frac{2}{2} + \cancel{\frac{1}{3}}$
- $k=2$ $\quad \frac{2}{2\times 3\times 4} = \frac{1}{2} - \cancel{\frac{2}{3}} + \cancel{\frac{1}{4}}$
- $k=3$ $\quad \frac{2}{3\times 4\times 5} = \cancel{\frac{1}{3}} - \cancel{\frac{2}{4}} + \cancel{\frac{1}{5}}$
- $k=4$ $\quad \frac{2}{4\times 5\times 6} = \cancel{\frac{1}{4}} - \cancel{\frac{2}{5}} + \cancel{\frac{1}{5}}$

$\vdots$

- $k=n-1$ $\quad \frac{2}{(n-1)n(n+1)} = \cancel{\frac{1}{n-1}} - \cancel{\frac{2}{n}} + \frac{1}{n+1}$
- $k=n$ $\quad \frac{2}{n(n+1)(n+2)} = \cancel{\frac{1}{n}} - \frac{2}{n+1} + \frac{1}{n+2}$

$$\Longrightarrow \sum_{k=1}^{n} \frac{2}{k(k+1)(k+2)} = \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}$$

$$\Longrightarrow 2\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{(n+1)(n+2) - 2(n+2) + 2(n+1)}{2(n+1)(n+2)}$$

IYGB - FP2 PAPER R - QUESTION 4

$$\Longrightarrow 2\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{n^2+3n\cancel{+2}\cancel{-2n}\cancel{-4}+\cancel{2n}+\cancel{2}}{2(n+1)(n+2)}$$

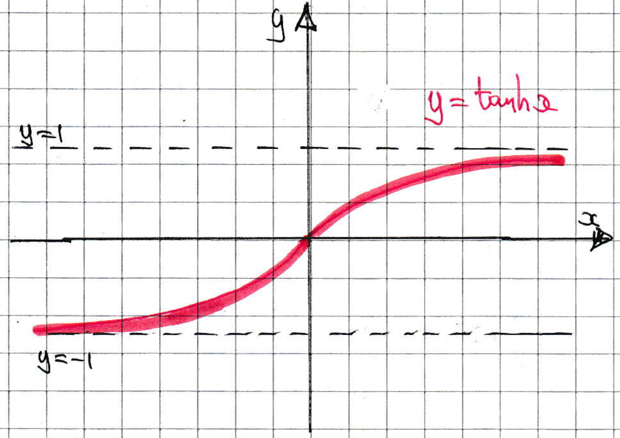
$$\Longrightarrow 2\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{n^2+3n}{2(n+1)(n+2)}$$

$$\Longrightarrow \sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

AS REQUIRED

IYGB - FP2 PAPER R - QUESTION 5

a) STARTING WITH THE GRAPH OF $y = \tanh x = \frac{e^{2x}-1}{e^{2x}+1}$



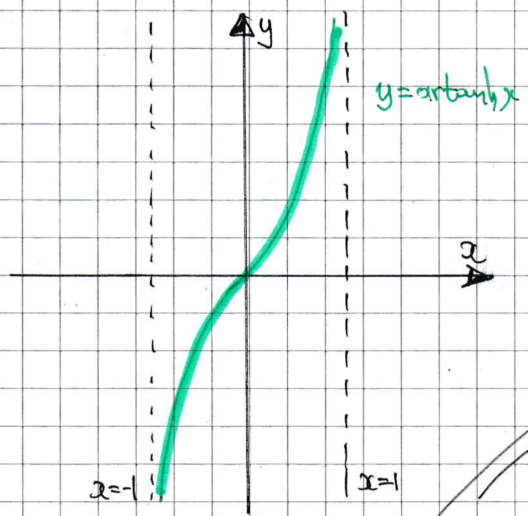
< ODD FUNCTION >

< PASSES THROUGH O >

< GRADIENT AT O IS 1 >

< ASYMPTOTES $y = \pm 1$ >

REFLECTING ABOUT $y = x$, GIVES THE GRAPH OF $y = \text{artanh}\, x$



b) PROCEED AS FOLLOWS

$$y = \text{artanh}\, x \implies \tanh y = x$$

$$\implies x = \tanh y$$

$$\implies \frac{dx}{dy} = \text{sech}^2 y$$

$$\implies \frac{dx}{dy} = 1 - \tanh^2 y$$

$$\implies \frac{dx}{dy} = 1 - x^2$$

$$\implies \frac{dy}{dx} = \frac{1}{1-x^2}$$

IYGB – FP2 PAPER R – QUESTION 5

c) USING PART (b)

If $y = \text{artanh}\,x \implies \frac{dy}{dx} = \frac{1}{1-x^2}$

$$\implies 1\,dy = \frac{1}{1-x^2}\,dx$$

INTEGRATE SUBJECT TO THE CONDITION $x=0, y=0$, SINCE $\text{artanh}\,0 = 0$

$$\implies \int_0^y 1\,dy = \int_0^x \frac{1}{1-x^2}\,dx$$

$$\implies \int_0^y 1\,dy = \int_0^x \frac{1}{(1+x)(1-x)}\,dx$$

PARTIAL FRACTIONS BY INSPECTION

$$\implies \int_0^y 1\,dy = \int_0^x \frac{\frac{1}{2}}{1+x} + \frac{\frac{1}{2}}{1-x}\,dx$$

$$\implies \Big[y\Big]_0^y = \Big[\tfrac{1}{2}\ln|1+x| - \tfrac{1}{2}\ln|1-x|\Big]_0^x$$

$$\implies y - 0 = \Big[\tfrac{1}{2}\ln\left|\tfrac{1+x}{1-x}\right|\Big]_0^x$$

$$\implies y = \tfrac{1}{2}\ln\left|\tfrac{1+x}{1-x}\right| - \tfrac{1}{2}\ln 1$$

$$\implies \text{artanh}\,x = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

IYGB - FP2 PAPER R - QUESTION 6

START BY FINDING THE CUBE ROOTS OF $-4+4\sqrt{3}i$

- $|-4+4\sqrt{3}i| = 4|-1+\sqrt{3}i| = 4\sqrt{1+3} = 8$
- $\arg(-4+4\sqrt{3}i) = \arg(-1+\sqrt{3}i) = \pi + \arctan\left(\frac{\sqrt{3}}{-1}\right)$
  $= \pi + \left(-\frac{\pi}{3}\right) = \frac{2\pi}{3}$

$$\Rightarrow z^3 = -4+4\sqrt{3}i$$
$$\Rightarrow z^3 = 8e^{\left(\frac{2\pi}{3}+2k\pi\right)i} \qquad k = 0,1,2$$
$$\Rightarrow z^3 = 8e^{\frac{2\pi}{3}i(1+3k)}$$
$$\Rightarrow z = \left[8e^{\frac{2\pi}{3}i(3k+1)}\right]^{\frac{1}{3}}$$
$$\Rightarrow z = 2e^{\frac{2\pi}{9}i(3k+1)}$$
$$\Rightarrow z = \begin{cases} 2e^{\frac{2\pi}{9}i} \\ 2e^{\frac{8\pi}{9}i} \\ 2e^{\frac{14\pi}{9}i} \end{cases}$$

NOW AS THE COEFFICIENT OF z^2 IS ZERO $\quad \alpha+\beta+\gamma = \text{"}-\frac{b}{a}\text{"} = 0$

$$\Rightarrow 2e^{\frac{2\pi}{9}i} + 2e^{\frac{8\pi}{9}i} + 2e^{\frac{14\pi}{9}i} = 0$$
$$\Rightarrow e^{\frac{2\pi}{9}i} + e^{\frac{8\pi}{9}i} + e^{\frac{14\pi}{9}i} = 0$$
$$\Rightarrow \left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right) + \left(\cos\frac{8\pi}{9} + i\sin\frac{8\pi}{9}\right) + \left(\cos\frac{14\pi}{9} + i\sin\frac{14\pi}{9}\right) = 0$$

LOOKING AT THE REAL PART

$$\Rightarrow \cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{8\pi}{9}\right) + \cos\left(\frac{14\pi}{9}\right) = 0$$

PERIODICITY

$$\Rightarrow \cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{8\pi}{9}\right) + \cos\left(\frac{14\pi}{9} - 2\pi\right) = 0$$
$$\Rightarrow \cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{8\pi}{9}\right) + \cos\left(-\frac{4\pi}{9}\right) = 0$$

EVEN FUNCTION

$$\Rightarrow \cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{8\pi}{9}\right) + \cos\left(\frac{4\pi}{9}\right) = 0$$

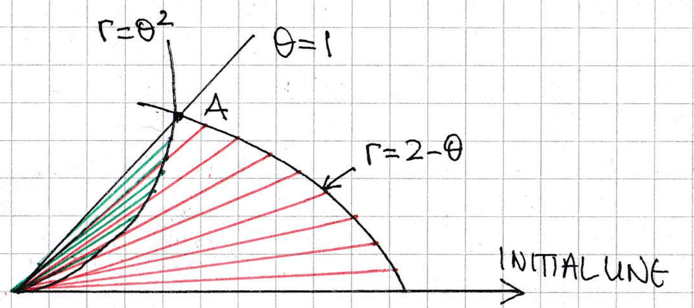
IYGB - FP2 PAPER R - QUESTION 7

a) SOLVING SIMULTANEOUSLY

$$\left.\begin{array}{l} r=\theta^2 \\ r=2-\theta \end{array}\right) \Longrightarrow \begin{array}{l} \theta^2 = 2-\theta \\ \theta^2+\theta-2=0 \\ (\theta+2)(\theta-1)=0 \\ \theta = \begin{cases} 1^c \\ \cancel{-2^c} \end{cases} \end{array}$$

$\therefore A(1, 1^c)$

b) DRAWING A DIAGRAM



"REQUIRED AREA" = "RED SECTORS" − "GREEN SECTORS"

$$\Longrightarrow \text{AREA} = \underbrace{\frac{1}{2}\int_{\theta=0}^{\theta=1} (2-\theta)^2 \, d\theta}_{\text{"RED SECTION"}} - \underbrace{\frac{1}{2}\int_{\theta=0}^{\theta=1} (\theta^2)^2 \, d\theta}_{\text{GREEN SECTION}}$$

$$\Longrightarrow \text{AREA} = \frac{1}{2}\int_0^1 (2-\theta)^2 - \theta^4 \, d\theta$$

$$= \frac{1}{2}\int_0^1 4-4\theta+\theta^2-\theta^4 \, d\theta$$

$$= \frac{1}{2}\left[4\theta - 2\theta^2 + \frac{1}{3}\theta^3 - \frac{1}{5}\theta^5\right]_0^1$$

$$= \frac{1}{2}\left[\left(4-2+\frac{1}{3}-\frac{1}{5}\right) - 0\right]$$

$$= \frac{16}{15}$$

AS REQUIRED

IYGB - FP2 PAPER R - QUESTION 8

a) USING THE SUBSTITUTION GIVEN

$$y = \frac{1}{x} \implies \frac{dy}{dx} = -\frac{1}{x^2}$$
$$\implies dx = -x^2\,dy$$
$$\implies dx = -\frac{1}{y^2}\,dy$$

$$\implies \int \frac{\ln x^2}{x^3}\,dx = \int \frac{2\ln x}{x^3}\,dx = \int (2\ln x)\left(\frac{1}{x^3}\right)dx$$
$$= \int 2\ln\left(\frac{1}{y}\right) \times y^3 \times \left(-\frac{1}{y^2}\,dy\right)$$
$$= \int -2y\ln\left(\frac{1}{y}\right)dy = \int -2y\ln y^{-1}\,dy$$
$$= \int 2y\ln y\,dy$$

AS REQUIRED

b) PROCEED BY INTEGRATION BY PARTS

$$\int_1^\infty \frac{\ln x^2}{x^3}\,dx = \ldots \text{ SUBSTITUTION FROM PART (a) } \ldots = \int_1^0 2y\ln y\,dy$$

$x = 1 \longmapsto y = 1$

$x = \infty \longmapsto y = 0$

BY PARTS

$\ln y$	$\frac{1}{y}$
y^2	$2y$

$$= \left[y^2\ln y\right]_1^0 - \int_1^0 y\,dy$$
$$= \left[y^2\ln y - \frac{1}{2}y^2\right]_1^0$$
$$= \left[\frac{1}{2}y^2 - y^2\ln y\right]_0^1$$

IYGB - FP2 PAPER R - QUESTION 8

APPLY LIMITS

$$\ldots = \lim_{h \to 0} \left[\frac{1}{2}y^2 - y^2 \ln y \right]_h^1$$

$$= \lim_{h \to 0} \left[\left(\frac{1}{2} - \cancel{\ln 1} \right) - \left(\frac{1}{2}h^2 - h^2 \ln h \right) \right]$$

NOW h^2 TENDS TO ZERO FASTER THAN $\ln h$ TENDS TO $-\infty$

$$= \lim_{h \to 0} \left[\frac{1}{2} - \cancel{\frac{1}{2}h^2} + \cancel{h^2 \ln h} \right]$$

$$= \frac{1}{2}$$