C4, IYGB, PAPER V

1. $$\int_1^e \ln x = \int_1^e 1 \times \ln x \, dx$$

$\ln x$	$\frac{1}{x}$
x	1

IGNORING LIMITS

$$x\ln x - \int 1 \, dx$$
$$= x\ln x - x + C$$

$$\ldots = \left[x\ln x - x\right]_1^e = (e\ln e - e) - (\cancel{1\ln 1} - 1)$$
$$= e - e + 1 = 1$$

2. a) $$(125 - 27x)^{\frac{1}{3}} = 125^{\frac{1}{3}}\left(1 - \frac{27}{125}x\right)^{\frac{1}{3}} = 5\left(1 - \frac{27}{125}x\right)^{\frac{1}{3}}$$
$$= 5\left[1 + \frac{\frac{1}{3}}{1}\left(-\frac{27}{125}x\right)^1 + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{1\times 2}\left(-\frac{27}{125}x\right)^2 + O(x^3)\right]$$
$$= 5\left[1 - \frac{9}{125}x - \frac{81}{15625}x^2 + O(x^3)\right]$$
$$= 5 - \frac{9}{25}x - \frac{81}{3125}x^2 + O(x^3)$$

b) $$\sqrt[3]{125 - 27x} \approx 5 - \frac{9}{25}x - \frac{81}{3125}x^2$$

$125 - 27x = 120$
$5 = 27x$
$x = \frac{5}{27}$

LET $x = \frac{5}{27}$

$$\sqrt[3]{120} \approx 5 - \frac{9}{25}\left(\frac{5}{27}\right) - \frac{81}{3125}\left(\frac{5}{27}\right)^2$$
$$\sqrt[3]{120} \approx 5 - \frac{1}{15} - \frac{1}{1125}$$
$$\sqrt[3]{120} \simeq \frac{5625 - 75 - 1}{1125} = \frac{5549}{1125}$$ AS REQUIRED

3. $x^3+y^3=6xy$

Diff w.r.t x

$\Rightarrow 3x^2+3y^2\frac{dy}{dx} = 6\times y + 6x\times\frac{dy}{dx}$

$\Rightarrow 3x^2+3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$

BUT $\frac{dy}{dx}=0$ (STATIONARY)

$\Rightarrow 3x^2 = 6y$

$\Rightarrow \boxed{y=\frac{1}{2}x^2}$

SOLVING SIMULTANEOUSLY WITH $x^3+y^3=6xy$

$\Rightarrow x^3+\left(\frac{1}{2}x^2\right)^3 = 6x\left(\frac{1}{2}x^2\right)$

$\Rightarrow x^3+\frac{1}{8}x^6 = 3x^3$

$\Rightarrow \frac{1}{8}x^6-2x^3=0$

$\Rightarrow \frac{1}{8}x^3\left(x^3-16\right)=0$

$x^3 = \begin{cases} \cancel{0} & \leftarrow \text{ORIGIN} \\ 16 & \leftarrow \text{"A"} \end{cases}$

AS ANSWERS MUST BE POWERS OF 2 WE WRITE AS

$x^3=16$

$\Rightarrow x^3=2^4$

$\Rightarrow \left(x^3\right)^{\frac{1}{3}} = \left(2^4\right)^{\frac{1}{3}}$

$\Rightarrow x = 2^{\frac{4}{3}}$

&

$y=\frac{1}{2}x^2$

$y=\frac{1}{2}\left(2^{\frac{4}{3}}\right)^2$

$y=\frac{1}{2}\times 2^{\frac{8}{3}}$

$y=2^{-1}\times 2^{8/3}$

$y=2^{5/3}$

$\therefore A\left(2^{\frac{4}{3}}, 2^{\frac{5}{3}}\right)$

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4. $\underline{r}_1 = (2,10,14) + \lambda(1,1,2) = (\lambda+2, \lambda+10, 2\lambda+14)$
$\underline{r}_2 = (a,8,4) + \mu(4,b,1) = (4\mu+a, \mu b+8, \mu+4)$

a) INTERSECTION TAKES PLACE AT $P(x,y,6)$

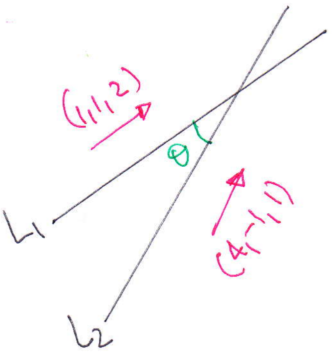
$\hat{k}$: $2\lambda+14=6$
$2\lambda=-8$
$\boxed{\lambda=-4}$

$\therefore P(-4+2, -4+10, 2(-4)+14)$
$P(-2,6,6)$

b) $\hat{k}$: $\mu+4=6$
$\boxed{\mu=2}$

$\hat{i}$: $4\mu+a=-2$
$4(2)+a=-2$
$8+a=-2$
$a=-10$

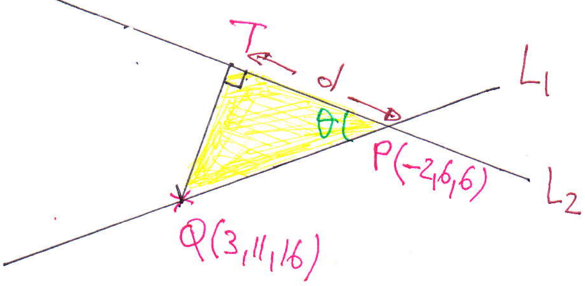
$\hat{j}$: $\mu b+8=6$
$2b+8=6$
$2b=-2$
$b=-1$

c) 

DOTTING DIRECTION VECTORS

$\Rightarrow (1,1,2)\cdot(4,-1,1) = |1,1,2|\,|4,-1,1|\cos\theta$
$\Rightarrow 4-1+2 = \sqrt{1+1+4}\,\sqrt{16+1+1}\cdot\cos\theta$
$\Rightarrow 5 = \sqrt{6}\sqrt{18}\cos\theta$
$\Rightarrow 5 = 6\sqrt{3}\cos\theta$
$\Rightarrow 5\sqrt{3} = 18\cos\theta$
$\Rightarrow \cos\theta = \frac{5\sqrt{3}}{18}$

d) $Q(3,11,16)$



- $\overrightarrow{PQ} = \underline{q} - \underline{p} = (3,11,16)-(-2,6,6)$
  $\overrightarrow{PQ} = (5,5,10)$
- $|PQ| = \sqrt{25+25+100} = \sqrt{150}$
- $\frac{|PT|}{|PQ|} = \cos\theta$
  $|PT| = |PQ|\cos\theta$
  $|PT| = \sqrt{150}\times\frac{5\sqrt{3}}{18} = \frac{25}{6}\sqrt{2}$

5. FIRST $y=0$ $\Rightarrow$ $\sin x + \cos x = 0$

$\Rightarrow$ $\sin x = -\cos x$

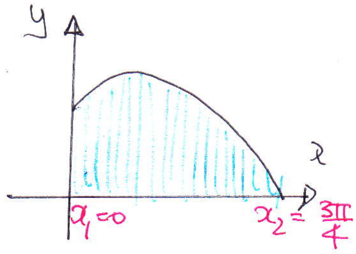
$\Rightarrow$ $\frac{\sin x}{\cos x} = -1$

$\Rightarrow$ $\boxed{\tan x = -1}$

$\arctan(-1) = -\frac{\pi}{4}$

$\Rightarrow$ $x = -\frac{\pi}{4} \pm n\pi$

$\boxed{x = \frac{3\pi}{4}}$ ← NEEDED (FROM THE GRAPH)



$$V = \pi \int_{x_1}^{x_2} \left(y(x)\right)^2 \, dx$$

$$V = \pi \int_0^{\frac{3\pi}{4}} (\sin x + \cos x)^2 \, dx = \pi \int_0^{\frac{3\pi}{4}} \sin^2 x + \underbrace{2\sin x \cos x}_{\sin 2x} + \cos^2 x \, dx$$

$$= \pi \int_0^{\frac{3\pi}{4}} 1 + \sin 2x \, dx$$

$$= \pi \left[x - \frac{1}{2}\cos 2x \right]_0^{\frac{3\pi}{4}}$$

$$= \pi \left[\left(\frac{3\pi}{4} - \cancel{\frac{1}{2}\cos\frac{3\pi}{2}} \right) - \left(0 - \frac{1}{2}\cos 0 \right) \right]$$

$$= \pi \left[\frac{3\pi}{4} + \frac{1}{2} \right]$$

$$= \frac{1}{4}\pi \left[3\pi + 2 \right]$$

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P = population (MILLIONS)
t = time (YEARS)
t=0, $P=\frac{1}{4}$

6. a)
$$\frac{dP}{dt} = P(1-P)$$
$$\Rightarrow \frac{1}{P(1-P)}\,dP = 1\,dt$$
$$\Rightarrow \int \frac{1}{P(1-P)}\,dP = \int 1\,dt$$

BY PARTIAL FRACTIONS

$$\frac{1}{P(1-P)} = \frac{A}{P} + \frac{B}{1-P}$$
$$\boxed{1 \equiv A(1-P) + BP}$$

If $P=1 \Rightarrow 1 = B$
If $P=0 \Rightarrow 1 = A$

$$\Rightarrow \int \frac{1}{P} + \frac{1}{1-P}\,dP = \int 1\,dt$$
$$\Rightarrow \ln|P| - \ln|1-P| = t + C$$
$$\Rightarrow \ln\left|\frac{P}{1-P}\right| = t + C$$
$$\Rightarrow \frac{P}{1-P} = e^{t+C}$$
$$\Rightarrow \frac{P}{1-P} = e^{t} \times e^{C} \leftarrow A$$
$$\Rightarrow \boxed{\frac{P}{1-P} = Ae^{t}}$$

• when $t=0$ $P=\frac{1}{4}$ $\Rightarrow$ $\frac{\frac{1}{4}}{1-\frac{1}{4}} = Ae^{0}$
$$\boxed{\frac{1}{3} = A}$$

$$\Rightarrow \frac{P}{1-P} = \frac{1}{3}e^{t}$$
$$\Rightarrow \frac{3P}{1-P} = e^{t}$$

AS REQUIRED

b) $\frac{3P}{1-P} = e^t$

$\Rightarrow 3P = (1-P)e^t$

$\Rightarrow 3P = e^t - Pe^t$

$\Rightarrow 3P + Pe^t = e^t$

$\Rightarrow P(3+e^t) = e^t$

$\Rightarrow P = \frac{e^t}{3+e^t}$

$\Rightarrow P = \frac{e^{-t}e^t}{3e^{-t} + e^t e^{-t}}$

$\Rightarrow P = \frac{e^0}{3e^{-t} + e^0}$

$\Rightarrow P = \frac{1}{1+3e^{-t}}$

AS REQUIRED

c) AS $t \to \infty$ $\quad e^{-t} \to 0$

$P \to \frac{1}{1+(3\times 0)}$

$P \to 1$

IE 1 MILLION

d) $\boxed{P = \frac{3}{4}}$

USING $e^t = \frac{3P}{1-P}$

$\Rightarrow e^t = \frac{3\times\frac{3}{4}}{1-\frac{3}{4}}$

$\Rightarrow e^t = 9$

$\Rightarrow t = \ln 9 = 2\ln 3$

$\Rightarrow t \approx 2.20$ (YEARS)

7. a) $x = t^2 - 9 \qquad y = t(4-t)^2$

• $y=0$

$t(4-t)^2 = 0$

$t = \begin{cases} 0 \\ 4 \end{cases} \quad x = \begin{cases} -9 \\ 7 \end{cases}$

$\therefore P(-9,0)$ & $Q(7,0)$

• $x=0$

$t^2 - 9 = 0$

$t = \begin{cases} 3 \\ -3 \end{cases} \quad y = \begin{cases} 3 \\ -147 \end{cases}$

$\therefore R(0,3) \quad T(0,-147)$

b)

$x = t^2 - 9 \qquad y = t(4-t)^2$

$\qquad\qquad\qquad\;\; y = t^3 - 8t^2 + 16t$

- $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-16t+16}{2t}$
- SOLVE FOR ZERO $\quad 3t^2 - 16t + 16 = 0$

$(3t-4)(t-4) = 0$

$t = 4 \leftarrow$ POINT Q

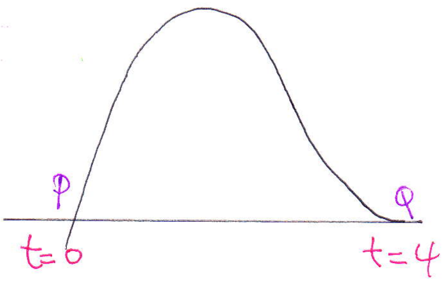
$t = \frac{4}{3} \leftarrow$ POINT S

$\therefore \quad x = \left(\frac{4}{3}\right)^2 - 9 = -65/9$

$y = \frac{4}{3}\left(4 - \frac{4}{3}\right)^2 = 256/27$

$S\left(-\frac{65}{9}, \frac{256}{27}\right)$

c)



$\Rightarrow A = \int_{x_1}^{x_2} y(x)\, dx$

$\Rightarrow A = \int_{t_1}^{t_2} y(t) \frac{dx}{dt}\, dt$

$\Rightarrow A = \int_0^4 \underbrace{(t^3 - 8t^2 + 16t)}_{y}\underbrace{(2t)}_{\frac{dx}{dt}}\, dt$

$\Rightarrow A = \int_0^4 2t^4 - 16t^3 + 32t^2\, dt$

$A = \left[\frac{2}{5}t^5 - 4t^4 + \frac{32}{3}t^3\right]_0^4$

$\Rightarrow A = \left(\frac{2048}{5} - 1024 + \frac{1024}{3}\right) - (0)$

$\Rightarrow A = \frac{1024}{15}$

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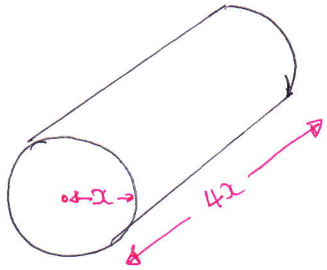
8.



- V = VOLUME
- A = CROSS SECTION

- $A = \pi x^2$
- $V = 4\pi x^3$

$$\frac{dA}{dt} = 0.036 \quad \text{(GIVEN)}$$

$$\Rightarrow \frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$

$$\Rightarrow \frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dA} \times \frac{dA}{dt}$$

- $A = \pi x^2, \ \frac{dA}{dx} = 2\pi x$
- $V = 4\pi x^3, \ \frac{dV}{dx} = 12\pi x^2$

$$\Rightarrow \frac{dV}{dt} = (12\cancel{\pi}x^2) \times \frac{1}{(2\cancel{\pi}x)} \times 0.036$$

$$\Rightarrow \frac{dV}{dt} = \frac{27}{125}x$$

$$\Rightarrow \left.\frac{dV}{dt}\right|_{x=1.25} = \frac{27}{125} \times 1.25 = 0.27 \ \text{cm}^3\,\text{s}^{-1}$$