C4, IYGB, PAPER Q

1. a) $\int 4xe^{2x} = \dots$ BY PARTS

$4x$	4
$\frac{1}{2}e^{2x}$	e^{2x}

$$\dots = 2xe^{2x} - \int 2e^{2x}\,dx$$

$$= 2xe^{2x} - e^{2x} + C$$

b) $$\int_0^{\ln 2} f(x)\,dx = \left[2xe^{2x} - e^{2x}\right]_0^{\ln 2}$$

$$= \left(2\ln 2\, e^{2\ln 2} - e^{2\ln 2}\right) - (0 - 1)$$

$$= 2\ln 2 \times 4 - 4 + 1$$

$$= -3 + 8\ln 2$$

2. a) $$(1+ax)^n = 1 + \frac{n}{1}(ax)^1 + \frac{n(n-1)}{1\times 2}(ax)^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3}(ax)^3 + O(x^4)$$

$$= 1 + \underbrace{\boxed{na}}_{2}x + \underbrace{\boxed{\tfrac{1}{2}n(n-1)a^2}}_{\frac{1}{2}}x^2 + \underbrace{\boxed{\tfrac{1}{6}n(n-1)(n-2)a^3}}_{b}x^3 + O(x^4)$$

SOLVING

$$\left.\begin{array}{l} na = 2 \\ \frac{1}{2}n(n-1)a^2 = \frac{1}{2}\end{array}\right\} \Rightarrow \left.\begin{array}{l} a = \frac{2}{n} \\ n(n-1)a^2 = 1\end{array}\right\} \Rightarrow \cancel{n}(n-1)\left(\frac{4}{n^{\cancel{2}}}\right) = 1$$

a^2

$$\Rightarrow \frac{4(n-1)}{n} = 1$$

$$\Rightarrow 4n - 4 = n$$

$$\Rightarrow 3n = 4$$

$$\Rightarrow n = \frac{4}{3}$$

& $a = \frac{2}{n} = \frac{2}{4/3} = \frac{3}{2}$

b) $$b = \frac{1}{6}n(n-1)(n-2)a^3 = \frac{1}{6}\left(\frac{4}{3}\right)\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(\frac{3}{2}\right)^3 = -\frac{1}{6}$$

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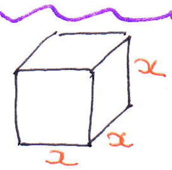
c) VALID FOR $|ax| < 1 \implies \left|\frac{3}{2}x\right| < 1$

$\implies |x| < \frac{2}{3}$

$-\frac{2}{3} < x < \frac{2}{3}$

3. $\boxed{\frac{dx}{dt} = 1.5 \text{, given}}$



$V = x^3$

$\frac{dV}{dx} = 3x^2$

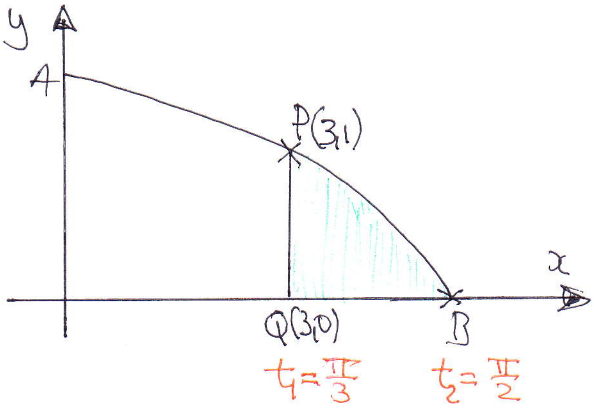
$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$

$\frac{dV}{dt} = 3x^2 \times 1.5$

$\frac{dV}{dt} = \frac{9}{2}x^2$

$\left.\frac{dV}{dt}\right|_{x=6} = \frac{9}{2} \times 6^2 = 162 \text{ cm}^3\text{s}^{-1}$

4. a)



$x = 4\sin^2 t$

$y = 2\cos t$

$0 \leq t \leq \frac{\pi}{2}$

- $x = 3$

$3 = 4\sin^2 t$

$\sin^2 t = \frac{3}{4}$

$\sin t = \pm\frac{\sqrt{3}}{2}$

$t = \frac{\pi}{3}$ (ONLY VALUE IN RANGE)

- $y = 0$

$0 = 2\cos t$

$\cos t = 0$

$t = \frac{\pi}{2}$ (ONLY VALUE IN RANGE)

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$$\text{AREA} = \int_{x_1}^{x_2} y(x)\,dx = \int_{t_1}^{t_2} y(t)\frac{dx}{dt}\,dt = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (2\cos t)(8\sin t\cos t)\,dt$$

$y(t)$ $\frac{dx}{dt}$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 16\sin t\cos^2 t = 16\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2 t\sin t\,dt$$

AS REQUIRED

b) BY INSPECTION (REVERSE CHAIN RULE)

$$\text{AREA} = 16\left[-\frac{1}{3}\cos^3 t\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{16}{3}\left[\cos^3 t\right]_{\pi/2}^{\pi/3}$$

$$= \frac{16}{3}\left[\frac{1}{8} - 0\right] = \frac{2}{3}$$

5. a) $2y^2 - xy + 4x + x^2 = 7$

Diff w.r.t x

$$4y\frac{dy}{dx} - 1 \times y - x\frac{dy}{dx} + 4 + 2x = 0$$

$$(4y - x)\frac{dy}{dx} = y - 2x - 4$$

$$\frac{dy}{dx} = \frac{y - 2x - 4}{4y - x}$$

b) $\frac{dy}{dx} = 0$ $\quad y - 2x - 4 = 0$

$$\boxed{y = 2x + 4}$$

SUB INTO THE EQUATION OF THE CURVE

$$\Rightarrow 2(2x+4)^2 - x(2x+4) + 4x + x^2 = 7$$

$\Rightarrow 8x^2+32x+32-2x^2-\cancel{4x}+\cancel{4x}+x^2=7$

$\Rightarrow 7x^2+32x+25=0$

$\Rightarrow (x+1)(7x+25)=0$

FROM $(-1,2)$

$\therefore x = \begin{cases} -1 \\ -\frac{25}{7} \end{cases} \qquad y = \begin{cases} 2 \\ -\frac{22}{7} \end{cases}$

$\therefore (-1,2)$ & $\left(-\frac{25}{7}, -\frac{22}{7}\right)$

6. a) $(2+\tan 3x)^2 = 4+4\tan 3x+\tan^2 3x$

$= 4+4\tan 3x+(\sec^2 3x-1)$ $\quad 1+\tan^2 A \equiv \sec^2 A$

$= 3+4\tan 3x+\sec^2 3x$

b) $\int \tan x \, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{-\sin x}{\cos x}\, dx$

NOW $\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)|+C$

$= -\ln|\cos x|+C = \ln\left|\frac{1}{\cos x}\right|+C$ $\quad \ln\left(\frac{a}{b}\right) = -\ln\left(\frac{b}{a}\right)$

$= \ln|\sec x|+C$

AS REQUIRED

P.T.O

c) $V = \pi \int_{x_1}^{x_2} (y(x))^2 \, dx = \pi \int_0^{\frac{\pi}{9}} (2+\tan 3x)^2 \, dx$

$= \pi \int_0^{\frac{\pi}{9}} 3 + 4\tan 3x + \sec^2 3x \, dx = \pi \left[3x + \frac{4}{3}\ln|\sec 3x| + \frac{1}{3}\tan 3x \right]_0^{\frac{\pi}{9}}$

$= \pi \left[\left(\frac{\pi}{3} + \frac{4}{3}\ln 2 + \frac{\sqrt{3}}{3} \right) - \left(0 + \cancel{\frac{4}{3}\ln 1} + 0 \right) \right]$

$= \frac{\pi}{3}\left[\pi + 4\ln 2 + \sqrt{3} \right]$

AS REQUIRED

7. a) $\vec{AE} = \underline{e} - \underline{a} = (25,0,6) - (10,20,6) = (15,-20,0)$

SCALE IT TO $(3,-4,0)$

$\underline{r} = (10,20,6) + \lambda(3,-4,0)$

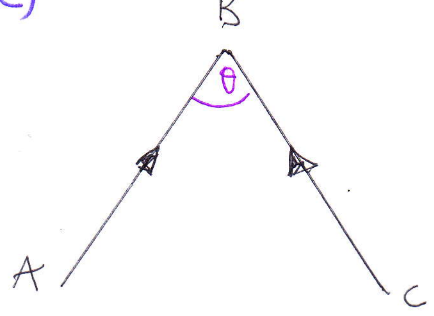
$\underline{r} = (3\lambda+10, 20-4\lambda, 6)$

b) $\vec{AC} = \underline{c} - \underline{a} = (2,14,6) - (10,20,6) = (-8,-6,0)$

$\vec{AE} \cdot \vec{AC} = (15,-20,0)\cdot(-8,-6,0) = -120 + 120 + 0 = 0$

INDEED PERPENDICULAR

c)



$\vec{AB} = \underline{b} - \underline{a} = (9,13,11) - (10,20,6) = (-1,-7,5)$

$\vec{CB} = \underline{b} - \underline{c} = (9,13,11) - (2,14,6) = (7,-1,5)$

$(-1,-7,5)\cdot(7,-1,5) = |-1,-7,5|\,|7,-1,5| \cos\theta$

$-7+7+25 = \sqrt{1+49+25}\,\sqrt{49+1+25}\,\cos\theta$

$25 = 75\cos\theta$

$\cos\theta = \frac{1}{3}$

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d) • "LINE AE": $\underline{r} = (10,20,6) + \lambda(3,-4,0)$

• LENGTH OF DIRECTION VECTOR $= |3,-4,0| = \sqrt{9+16+0} = \underline{5}$

• $10 \div 5 = 2$ ← "TWO DIRECTION VECTOR'S WORTH"

• THUS $\underline{d} = \underline{b} + 2 \times$ DIRECTION VECTOR

$\underline{d} = (9,13,11) + 2(3,-4,0)$

$\underline{d} = (15,5,11)$

$\therefore D(15,5,11)$

8. a) $\frac{dm}{dt} = k(m-6)(m-3)$

$\Rightarrow \frac{1}{(m-6)(m-3)}\,dm = k\,dt$

$\Rightarrow \int \frac{1}{(m-6)(m-3)}\,dm = \int k\,dt$

PARTIAL FRACTIONS

$\frac{1}{(m-6)(m-3)} \equiv \frac{A}{m-6} + \frac{B}{m-3}$

$\boxed{1 \equiv A(m-3) + B(m-6)}$

If $m=3$, $1 = -3B \Rightarrow B = -\frac{1}{3}$

If $m=6$, $1 = 3A \Rightarrow A = \frac{1}{3}$

$\Rightarrow \int \frac{\frac{1}{3}}{m-6} - \frac{\frac{1}{3}}{m-3}\,dm = \int k\,dt$

$\Rightarrow \int \frac{1}{m-6} - \frac{1}{m-3}\,dm = \int 3k\,dt$

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$$\Rightarrow \ln|m-6| - \ln|m-3| = 3kt + C$$

$$\Rightarrow \ln\left|\frac{m-6}{m-3}\right| = 3kt + C$$

$$\Rightarrow \frac{m-6}{m-3} = e^{3kt+C}$$

$$\Rightarrow \frac{m-6}{m-3} = e^{3kt} \times e^{C}$$

$$\Rightarrow \frac{m-6}{m-3} = Ae^{3kt} \quad \text{WHERE } A = e^{C}.$$

AS REQUIRED

b) NOW $t=0$, $m=0$

$$\frac{-6}{-3} = Ae^{0}$$

$$A = 2$$

$$\therefore \frac{m-6}{m-3} = 2e^{3kt}$$

WHEN $t = \ln 16$, $m = 2$

$$\Rightarrow \frac{2-6}{2-3} = 2e^{3k\ln 16}$$

$$\Rightarrow 4 = 2e^{3k\ln 16}.$$

$$\Rightarrow 2 = e^{3k\ln 16}$$

$$\Rightarrow \ln 2 = 3k\ln 16$$

$$\Rightarrow k = \frac{\ln 2}{3\ln 16}$$

$$\Rightarrow k = \frac{1}{12}$$

$$\therefore \frac{m-6}{m-3} = 2e^{\frac{1}{4}t}$$

THUS $m-6 = 2e^{\frac{1}{4}t}(m-3)$

$$\Rightarrow m-6 = 2me^{\frac{1}{4}t} - 6e^{\frac{1}{4}t}$$

$$\Rightarrow 6e^{\frac{1}{4}t} - 6 = 2me^{\frac{1}{4}t} - m$$

$$\Rightarrow m(2e^{\frac{1}{4}t} - 1) = 6e^{\frac{1}{4}t} - 6$$

$$\Rightarrow m = \frac{6e^{\frac{1}{4}t} - 6}{2e^{\frac{1}{4}t} - 1}$$

MULTIPLY TOP & BOTTOM BY $e^{-\frac{1}{4}t}$

GIVES

$$\Rightarrow m = \frac{6 - 6e^{-\frac{1}{4}t}}{2 - e^{-\frac{1}{4}t}}$$

AS REQUIRED

c) AS $t \to \infty$ $e^{-\frac{1}{4}t} \to 0$

$$m \to \frac{6}{2} = 3$$

IE 3 IS A LIMITING VALUE