C4, IYGB, PAPER N

1. $\int_0^1 5x(1-x^2)^{\frac{3}{2}}\,dx = \dots$ BY RECOGNITION OF THE CHAIN RULE

$= \left[-(1-x^2)^{\frac{5}{2}}\right]_0^1 = \left[(1-x^2)^{\frac{5}{2}}\right]_1^0 = 1^{\frac{5}{2}} - \cancel{0^{\frac{5}{2}}} = 1$

OR BY SUBSTITUTION

$\int_0^1 5x(1-x^2)^{\frac{3}{2}}\,dx = \dots$

$= \int_1^0 5\cancel{x}\,u^3\left(-\frac{u}{\cancel{x}}\,du\right)$

$= \int_1^0 -5u^4\,du = \int_0^1 5u^4\,du$

$= \left[u^5\right]_0^1 = 1^5 - 0^5 = 1$ AS BEFORE

$u^2 = 1-x^2$

$u = (1-x^2)^{\frac{1}{2}}$

$\cancel{2}u\frac{du}{dx} = -\cancel{2}x$

$-\frac{u\,du}{x} = dx$

$x=0 \quad u=1$

$x=1 \quad u=0$

2. a)
- $2\sqrt{1+4x} = 2(1+4x)^{\frac{1}{2}}$

$= 2\left[1 + \frac{\frac{1}{2}}{1}(4x)^1 + \frac{\frac{1}{2}(-\frac{1}{2})}{1\times2}(4x)^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1\times2\times3}(4x)^3 + O(x^4)\right]$

$= 2\left[1 + 2x - 2x^2 + 4x^3 + O(x^4)\right]$

$= 2 + 4x - 4x^2 + 8x^3 + O(x^4)$

- $\frac{4}{1+x} = 4(1+x)^{-1}$

$= 4\left[1 + \frac{-1}{1}(x)^1 + \frac{-1(-2)}{1\times2}(x)^2 + \frac{-1(-2)(-3)}{1\times2\times3}(x)^3 + O(x^4)\right]$

$= 4\left[1 - x + x^2 - x^3 + O(x^4)\right]$

$= 4 - 4x + 4x^2 - 4x^3 + O(x^4)$

$\therefore f(x) = \begin{array}{l} 2 + \cancel{4x} - \cancel{4x^2} + 8x^3 + O(x^4) \\ 4 - \cancel{4x} + \cancel{4x^2} - 4x^3 + O(x^4) \\ \hline 6 \qquad\qquad + 4x^3 + O(x^4) \end{array}$

$\therefore f(x) \simeq 6 + 4x^3$

b) VALIDITY IS THE "TIGHTEST" OUT OF $|4x|<1$ & $|x|<1$

$$|4x|<1 \Rightarrow |x|<\frac{1}{4}$$

$$-\frac{1}{4}<x<\frac{1}{4}$$

3. a) $x = 2t^2 + t^{-1}$ & $y = 2t^2 - t^{-1}$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t + t^{-2}}{4t - t^{-2}} = \frac{4t + \frac{1}{t^2}}{4t - \frac{1}{t^2}}$$

MULTIPLY TOP & BOTTOM BY t^2

$$\frac{dy}{dx} = \frac{4t^3+1}{4t^3-1}$$

$$\left.\frac{dy}{dx}\right|_{t=\frac{1}{2}} = \frac{4\left(\frac{1}{2}\right)^3+1}{4\left(\frac{1}{2}\right)^3-1} = \frac{\frac{1}{2}+1}{\frac{1}{2}-1} = \frac{3/2}{-1/2} = -3$$

b) $x+y = \left(2t^2+\cancel{\frac{1}{t}}\right)+\left(2t^2-\cancel{\frac{1}{t}}\right) = 4t^2$

$x-y = \left(\cancel{2t^2}+\frac{1}{t}\right)-\left(\cancel{2t^2}-\frac{1}{t}\right) = \frac{2}{t}$

THUS $(x+y)(x-y)^2 = 4t^2 \times \left(\frac{2}{t}\right)^2$

$(x+y)(x-y) = 4\cancel{t^2} \times \frac{4}{\cancel{t^2}}$

$(x+y)(x-y) = 16$

AS REQUIRED

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4.

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$\frac{dy}{dt} = 2e^{\frac{1}{5}x-1} \times \frac{3}{(6t+1)^{\frac{1}{2}}}$$

$$\frac{dy}{dt} = 2e^{\frac{1}{5}(6t+1)^{\frac{1}{2}}-1} \times \frac{3}{(6t+1)^{\frac{1}{2}}}$$

$$\left.\frac{dy}{dt}\right|_{t=4} = 2e^{0} \times \frac{3}{5} = \frac{6}{5}$$

- $y = 10e^{\frac{1}{5}x-1}$

$$\frac{dy}{dx} = 2e^{\frac{1}{5}x-1}$$

- $x = (6t+1)^{\frac{1}{2}}$

$$\frac{dx}{dt} = \frac{1}{2}(6t+1)^{-\frac{1}{2}} \times 6$$

$$\frac{dx}{dt} = \frac{3}{(6t+1)^{\frac{1}{2}}}$$

5. a)

$$y = \frac{2x+1}{xy+3}$$

$$\Rightarrow xy^2 + 3y = 2x + 1$$

Diff w.r.t x

$$\Rightarrow 1 \times y^2 + x\left(2y\frac{dy}{dx}\right) + 3\frac{dy}{dx} = 2$$

$$\Rightarrow 2xy\frac{dy}{dx} + 3\frac{dy}{dx} = 2 - y^2$$

$$\Rightarrow (2xy+3)\frac{dy}{dx} = 2 - y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{2-y^2}{2xy+3}$$

b) PARALLEL TO y AXIS $\Rightarrow$ VERTICAL

$\therefore$ INFINITE GRADIENT

$\therefore \frac{dy}{dx} = \infty$

$\therefore \boxed{2xy + 3 = 0}$

SOLVE SIMULTANEOUSLY WITH THE CURVE

$$\left(\begin{array}{l} xy^2 + 3y = 2x + 1 \\ x = -\frac{3}{2y} \end{array}\right) \Rightarrow$$

$$\Rightarrow -\frac{3}{2y} \times y^2 + 3y = 2\left(-\frac{3}{2y}\right) + 1$$

$$\Rightarrow -\frac{3}{2}y + 3y = -\frac{3}{y} + 1$$

$$\Rightarrow -3y + 6y = -\frac{6}{y} + 2$$

$$\Rightarrow 3y = -\frac{6}{y} + 2$$

$\Longrightarrow 3y^2 = -6 + 2y$

$\Longrightarrow 3y^2 - 2y + 6 = 0$

$b^2 - 4ac = (-2)^2 - 4\times3\times6 = 4 - 72 = -68 < 0$

$\therefore$ NO SOLUTIONS $\Rightarrow$ GRADIENT IS NEVER INFINITY

6. a) $\underline{a} = (5,-1,-1)$

$\underline{b} = (1,-5,7)$

$\vec{AB} = \underline{b} - \underline{a} = (1,-5,7) - (5,-1,-1) = (-4,-4,8)$

SCALE DIRECTION $(1,1,-2)$

$\underline{r} = (5,-1,-1) + \lambda(1,1,-2)$

$\underline{r} = (\lambda+5, \lambda-1, -2\lambda-1)$

b) (i) $\lambda + 5 = 4$ $\quad \boxed{\lambda = -1}$

(j) $\lambda - 1 = -2$ $\quad \boxed{\lambda = -1}$

(k) $-2\lambda - 1 = 1$

$-2\lambda = 2$

$\boxed{\lambda = -1}$

$\therefore$ C IS ON l

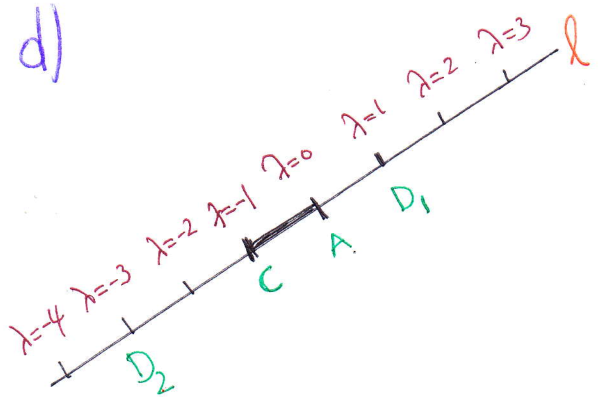
c) $\vec{OC} = \underline{c} = (4,-2,1)$

l HAS DIRECTION $(1,1,-2)$

$(4,-2,1)\cdot(1,1,-2) = 4-2-2 = 0$

PERPENDICULAR INDEED

d)



BY INSPECTION (DIAGRAM OPPOSITE)

$\lambda = 1$ OR $\lambda = -3$

$D(6,0,-3)$

OR $D(2,-4,5)$

ALTERNATIVE

LET $\underline{d} = (x, y, z)$, WHERE $x = \lambda + 5,\ y = \lambda - 1,\ z = -2\lambda - 1$

$|\overrightarrow{CD}| = 2|\overrightarrow{CA}|$

$\Rightarrow |\underline{d} - \underline{c}| = 2|\underline{a} - \underline{c}|$

$\Rightarrow |(x, y, z) - (4, -2, 1)| = 2|(5, -1, -1) - (4, -2, 1)|$

$\Rightarrow |x - 4,\ y + 2,\ z - 1| = 2|1, 1, -2|$

$\Rightarrow \sqrt{(x-4)^2 + (y+2)^2 + (z-1)^2} = 2\sqrt{1+1+4}$

$\Rightarrow (x-4)^2 + (y+2)^2 + (z-1)^2 = 24$

$\Rightarrow (\lambda + 5 - 4)^2 + (\lambda - 1 + 2)^2 + (-2\lambda - 1 - 1)^2 = 24$

$\Rightarrow (\lambda + 1)^2 + (\lambda + 1)^2 + (-2\lambda - 2)^2 = 24$

$$\Rightarrow \left\{\begin{array}{l} \lambda^2 + 2\lambda + 1 \\ \lambda^2 + 2\lambda + 1 \\ 4\lambda^2 + 8\lambda + 4 \end{array}\right\} = 24$$

$\Rightarrow 6\lambda^2 + 12\lambda - 18 = 0$

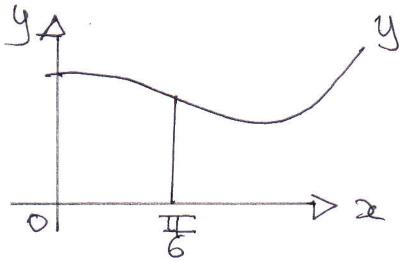
$\Rightarrow \lambda^2 + 2\lambda - 3 = 0$

$\Rightarrow (\lambda + 3)(\lambda - 1) = 0$

$\lambda = \begin{cases} -3 \\ 1 \end{cases} \Rightarrow \begin{array}{l} D(2, -4, 5) \\ D(6, 0, -3) \end{array}$

AS BEFORE

7.



$$\boxed{V = \pi \int_{x_1}^{x_2} (y(x))^2 \, dx}$$

$$V = \pi \int_0^{\frac{\pi}{6}} (\sec x + 4\cos x)^2 \, dx = \pi \int_0^{\frac{\pi}{6}} \sec^2 x + 8\sec x \cos x + 16\cos^2 x \, dx$$

$$\Rightarrow V = \pi \int_0^{\frac{\pi}{6}} \sec^2 x + 8 + 16\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right) dx$$

$$\Rightarrow V = \pi \int_0^{\frac{\pi}{6}} \sec^2 x + 16 + 8\cos 2x \, dx$$

$$\Rightarrow V = \pi \Big[\tan x + 16x + 4\sin 2x\Big]_0^{\frac{\pi}{6}}$$

$$\Rightarrow V = \pi \left[\left(\frac{\sqrt{3}}{3} + \frac{8}{3}\pi + 4 \times \frac{\sqrt{3}}{2}\right) - (0 + 0 + 0)\right]$$

$$\Rightarrow V = \pi \left[\tfrac{1}{3}\sqrt{3} + \tfrac{8}{3}\pi + 2\sqrt{3}\right]$$

$$\Rightarrow V = \pi \left[\tfrac{7}{3}\sqrt{3} + \tfrac{8}{3}\pi\right]$$

$$\Rightarrow V = \tfrac{1}{3}\pi \left[7\sqrt{3} + 8\pi\right]$$

AS REQUIRED

8. a)

$$\boxed{\frac{dP}{dt} = \frac{1}{20} P(2P - 1)\cos t \qquad \text{when } t = 0,\ P = 8}$$

SEPARATE VARIABLES

$$\Rightarrow \frac{20}{P(2P-1)} \, dP = \cos t \, dt$$

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$$\Longrightarrow \int \frac{20}{P(2P-1)}\,dP = \int \cos t\,dt$$

PARTIAL FRACTIONS

$$\frac{20}{P(2P-1)} \equiv \frac{A}{P} + \frac{B}{2P-1}$$

$$\boxed{20 \equiv A(2P-1) + BP}$$

If $P=0 \Longrightarrow 20 = -A$

If $P=\frac{1}{2} \Longrightarrow 20 = \frac{1}{2}B$

$\therefore A = -20$

$B = 40$

$$\Longrightarrow \int \frac{40}{2P-1} - \frac{20}{P}\,dP = \int \cos t\,dt$$

$$\Longrightarrow 20\ln|2P-1| - 20\ln|P| = \sin t + C$$

$$\Longrightarrow 20\ln\left|\frac{2P-1}{P}\right| = \sin t + C$$

$$\Longrightarrow \ln\left|\frac{2P-1}{P}\right| = \frac{1}{20}\sin t + C$$

$$\Longrightarrow \frac{2P-1}{P} = e^{\frac{1}{20}\sin t + C} = e^{\frac{1}{20}\sin t} \times e^{C} \leftarrow A$$

$$\Longrightarrow \boxed{\frac{2P-1}{P} = Ae^{\frac{1}{20}\sin t}}$$

$t=0 \quad P=8 \Longrightarrow \frac{15}{8} = Ae^{0}$

$A = \frac{15}{8}$

$$\Longrightarrow \frac{2P-1}{P} = \frac{15}{8}e^{\frac{1}{20}\sin t}$$

$$\Longrightarrow 16P - 8 = 15Pe^{\frac{1}{20}\sin t}$$

$$\Longrightarrow 16P - 15Pe^{\frac{1}{20}\sin t} = 8$$

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$$\Rightarrow P\left(16 - 15e^{\frac{1}{20}\sin t}\right) = 8$$

$$\Rightarrow P = \frac{8}{16 - 15e^{\frac{1}{20}\sin t}}$$

AS REQUIRED

b) $-1 \le \sin t \le 1$

- IF $\sin t = 1$

$$P = \frac{8}{16 - 15e^{\frac{1}{20}}} = 34.64199\ldots \quad \text{IE} \quad 34642$$

- IF $\sin t = -1$

$$P = \frac{8}{16 - 15e^{-\frac{1}{20}}} = 4.62011\ldots \quad \text{IE} \quad 4620$$

$\therefore$ MIN IS 4620

MAX IS 34600

3 s.f.